We have seen examples of chains of normal subgroups:

\[(1.1) \quad G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \cdots G_r = \{e\}\]

in which each group \(G_{i+1}\) is normal in the preceding group \(G_i\) (though not necessarily normal in \(G\)). Such a series is often called \textit{subnormal}, and this is the terminology we use. For example, there is the sequence of derived subgroups

\[G \supseteq D(G) = [G, G] \supseteq D^2(G) = [D(G), D(G)] \cdots\]

which ends with \(D^r(G) = \{e\}\) if \(G\) is a solvable group, in which \(D^i(G)/D^{i+1}(G)\) is abelian.

At the other extreme, the group \(G\) is \textit{simple} if it contains no proper normal subgroups other than \(\{e\}\). A subnormal series such as \((1.1)\) is called a \textit{composition series} if each of the quotient groups \(G_i/G_{i+1}\) is simple; in particular, \(G_i \neq G_{i+1}\) for all \(i\).

\textbf{Lemma 1.2.} \textit{Let} \(G\) \textit{be a finite group. Then} \(G\) \textit{has a composition series.}

\textit{Proof.} \textit{We induct on the order of} \(G\). \textit{We know that a group of order 1 has a composition series. Suppose every group of order less than} \(|G|\) \textit{has a composition series. If} \(G\) \textit{is simple, then we are done. If not, then} \(G\) \textit{has a non-trivial proper normal subgroup} \(N\). \textit{By induction,} \(N\) \textit{and} \(G/N\) \textit{both have composition series. Say}

\[G/N = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_r = \{e\}\]

\textit{is a composition series. By the correspondence principle, each} \(H_i\) \textit{corresponds to a subgroup} \(G_i\) \textit{containing} \(N\), \textit{with} \(H_i = G_i/N\) \textit{for all} \(i\). \textit{By the Third Isomorphism Theorem,}

\[G_i/G_{i+1} \xrightarrow{\sim} (G_i/N)/(G_{i+1}/N) = H_i/H_{i+1}\]

\textit{which is simple. On the other hand,} \(H_r = N\) \textit{has a composition series}

\[N = G_r \supseteq G_{r+1} \cdots \supseteq G_N = \{e\}.\]

\textit{Then}

\[G = G_0 \supseteq G_1 \cdots \supseteq N = G_r \supseteq G_{r+1} \cdots \supseteq G_N = \{e\}\]

\textit{is a composition series for} \(G\). \(\square\)
We write the collection of simple factors \((J_\alpha, m_\alpha)\) where \(J_\alpha\) is a simple group and \(m_\alpha\) is the number of times it appears as a quotient \(G_i/G_{i+1}\). We call \(m_\alpha\) the multiplicity of the simple factor \(J_\alpha\). We call it a collection rather than a set, because the same element can appear more than once; sometimes this is called a multiset.

**Example 1.3** (Cyclic groups of prime power order). The cyclic group \(\mathbb{Z}_{p^a}\) has a composition series:

\[
\mathbb{Z}_{p^a} \supseteq (p) \supseteq (p^2) \supseteq \cdots \supseteq (p^{a-1}) \supseteq \{0\}
\]

where \((p^i)\) denotes the multiples of \(p^i\) modulo \(p^a\), for any \(i \leq a\). We can use the Third Isomorphism Theorem: if \(\langle p^i \rangle \subseteq \mathbb{Z}\) is the subgroup of multiples of \(p^i\) for each \(i\), then the subgroups of \(\mathbb{Z}_{p^a}\) correspond to subgroups of \(\mathbb{Z}\) containing \(\langle p^a \rangle\). In particular

\[
(p^i) = \langle p^i \rangle / \langle p^a \rangle \subseteq \mathbb{Z}_{p^a}.
\]

Then by the Third Isomorphism Theorem

\[
(p^i) / (p^{i+1}) = (\langle p^i \rangle / \langle p^a \rangle) / (\langle p^{i+1} \rangle / \langle p^a \rangle) \cong (p^i) / (p^{i+1})
\]

and multiplication by \(p^i\) is an isomorphism

\[
\mathbb{Z} / \langle p \rangle \cong (p^i) / (p^{i+1}).
\]

So the collection of simple factors of \(\mathbb{Z}_{p^a}\) is \((\mathbb{Z}_p, a)\) (multiplicity \(a\)).

**Example 1.4** (Cyclic groups). Let \(n \in \mathbb{Z}\). Write \(n = \prod_i p_i^{a_i}\) as a product of prime factors. Then the cyclic group \(\mathbb{Z}_n\) is isomorphic to a product of cyclic groups \(\mathbb{Z}_{p_i^{a_i}}\) and the collection of simple factors of \(\mathbb{Z}_n\) is the union of the simple factors of all the \(\mathbb{Z}_{p_i^{a_i}}\):

\[
(\mathbb{Z}_{p_i^{a_i}}, a_i).
\]

**Example 1.5** (Abelian groups). We know that any abelian group is isomorphic to a direct product of cyclic groups:

\[
\prod_i \prod_j \mathbb{Z}_{p_i^{a_{ij}}}
\]

where the \(p_i\) are distinct prime numbers and the \(a_{ij}\) are positive integers. The only simple abelian groups are the cyclic groups of prime order. So the collection of simple factors is

\[
\{(\mathbb{Z}_{p_i}, m_i = \sum_j a_{ij})\}.
\]

In other words, \(\mathbb{Z}_{p_i}\) occurs as a simple factor \(a_{ij}\) times in the cyclic group \(\mathbb{Z}_{p_i^{a_{ij}}}\), and the total multiplicity is the sum of the multiplicities in the simple factors.

A given finite group \(G\) can have more than one composition series. Nevertheless, there is a uniqueness theorem that is analogous to the uniqueness of prime factorization of an integer. First we prove a lemma.
Lemma 1.6. Let $G$ be a group with two normal subgroups $H$ and $J$, $H \neq J$. Suppose $G/H$ and $G/J$ are both simple. Then

$$G/H \sim J/H \cap J; G/J \sim H/H \cap J.$$

Proof. If $H \subseteq J$ then $G/H \supseteq J/H$, and since $G/H$ is simple this implies $J = G$ or $J = H$, both of which are impossible. Thus $G \supseteq H \cdot J \supseteq J$ and since $H \cdot J \neq J$ we must have $G = H \cdot J$.

Now we apply the Second Isomorphism Theorem:

$$G/H = H \cdot J/H \sim J/H \cap J.$$

The same proof works for $G/J$. \qed

Theorem 1.7 (Jordan-Hölder Theorem). Let $G$ be a finite group. Suppose $G$ has two composition series:

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \cdots \supseteq G_{r+1} = \{e\}$$

$$G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{s+1} = \{e\}.$$

Then $r = s$ and the two collections of quotients

$$\{G_i/G_{i+1}\}, \{H_j/H_{j+1}\}$$

are equal (not taking order into account).

Proof. This is of course an induction proof. The case $|G| = 1$ is trivial. If $r = 1$ then $G$ is simple so again we must have $H_1 = G_1$. Now suppose the theorem is known for groups of order $|G|$. We assume $r$ is the minimal length of a composition series for $G$. Suppose $G_1 = H_1$. Then by induction on $|G|$ the composition series for $G_1$ and $H_1$ are equivalent, and so we are done. Thus we must assume $G_1 \neq H_1$. Now $G/G_1$ is simple, so the only subgroups of $G$ containing $G_1$ are $G_1$ and $G$. Since $G_1 \cdot H_1$ is normal in $G$ and contains but is not equal to $G_1$, we have $G = G_1 \cdot H_1$. Let $K_1 = G_1 \cap H_1$.

By the lemma,

$$G/G_1 \sim H_1/K_1; G/H_1 \sim G_1/K_1.$$ \(\text{Theorem 1.6} \)

Also $H_1 \subseteq H_0 = G_0$, the intersection $K_i = H_1 \cap G_i$ is normal in each $G_i$ as well. Also, $K_{i+1} \subseteq K_i$ for each $i$. So we have a new subnormal series

$$G = G_0 \supseteq G_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}.$$

Note that this is of length $r+1$. The image of $K_i$ in $G_i/G_{i+1}$ is normal and is isomorphic to

$$K_i/K_i \cap G_{i+1} = K_i/(H_1 \cap G_i) \cap G_{i+1} = K_i/H_1 \cap G_{i+1} = K_i/K_{i+1}$$

because $G_{i+1} \subseteq G_i$. Since $G_i/G_{i+1}$ is simple, we have either $K_i/K_{i+1} = G_i/G_{i+1}$ or $K_i = K_{i+1}$. In particular, every non-trivial term in

$$G_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}$$

is simple, and thus it must be a composition series when the non-trivial terms are removed. But by induction, every composition series for $G_1$ has
length \( r - 1 \) and any two are equivalent. So exactly one quotient \( K_j/K_{j+1} \) is trivial, and with that removed we have a composition series equivalent to
\[
G_1 \supseteq G_2 \supseteq \cdots \supseteq G_r \supseteq G_{r+1} = \{e\}.
\]
In particular the two collections
\[
\{K_i/K_{i+1}, i \neq j, i \geq 1\}
\]
and
\[
\{G_i/G_{i+1}, i \geq 1\}
\]
are the same.

On the other hand, we also have two composition series for \( H_1 \):
\[
H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{s+1} = \{e\}
\]
and
\[
H_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}
\]
where we remove the term \( K_j = K_{j+1} \). Again by induction, these are equivalent, but the first is of length \( s - 1 \) and the second of length \( r - 1 \). Thus \( r = s \). And again the two collections
\[
\{K_i/K_{i+1}, i \neq j, i \geq 1\}
\]
and
\[
\{H_i/H_{i+1}, i \geq 1\}
\]
are the same. So for \( i \geq 1 \), we have
\[
\{G_i/G_{i+1}, i \geq 1\} = \{H_i/H_{i+1}, i \geq 1\}
\]
Finally, we return to the two series
\[
G \supseteq H_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}
\]
and
\[
G \supseteq G_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}
\]
(with \( K_j \) omitted). By the lemma, as we have said, the two unordered pairs
\[
(G/H_1, H_1/K_1); (G/G_1, G_1/K_1)
\]
are equal. Moreover, the remaining terms are equal, so we are done. □