Homework 3, due February 13: Basic properties of groups

1. Let $X$ be a set with two elements $e, f$.
   (a) Can you define a binary operation
   $\star : X \times X \to X$
   that is not associative?
   (b) Suppose $e$ is a two-sided identity for $\star$, in other words
   $e \star e = e, \ e \star f = f \star e = f$.
   (Here we write $e \star e$ instead of $\star(e, e)$, as usual.) How many such operations
   are there? Are they all necessarily commutative? Associative?

2. (a) Let $(X, \star)$ and $(Y, \circ)$ be two sets with binary operations. Suppose
   $f : X \to Y$
is a bijection that defines an isomorphism of binary structures, i.e.
   $f(x_1 \star x_2) = f(x_1) \circ f(x_2)$.
   Show that $f^{-1} : Y \to X$ is also an isomorphism of binary structures.
   (b) In the notation of (a), if $X = Y$ and $\star = \circ$, show that the identity
   map from $X$ to itself defines an isomorphism of binary structures.

3. Let $n \geq 3$ be an integer. Let $\Delta_n$ be a regular polygon with $n$ sides in
   the complex plane, with one vertex at the point 1 and the other vertices on
   the circle $x^2 + y^2 = 1$. Let $\mu_n$ denote the set of vertices of $\Delta_n$.
   (a) Use either the exponential function or trigonometric functions to list
   the coordinates of the points in $\mu_n$.
   (b) Show that the subset $\mu_n \subset \mathbb{C}$ is a group under multiplication.
   (c) Define an isomorphism of groups $f : \mathbb{Z}/n\mathbb{Z} \to \mu_n$.
   (d) How many solutions does part (c) have? Explain.

4. List all subgroups of the Klein 4 group and of the cyclic group $\mathbb{Z}/4\mathbb{Z}$.
   How many subgroups contain 3 elements in each case?

5. Let $X$ be a set with 3 elements. How many distinct binary operations
   $X \times X \to X$
   are there?

6. A $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is idempotent if $A^2 = A$. 
(a) Check that the matrices \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) are both idempotents (you don’t need to write this down). Find an idempotent matrix that is equal to neither of these.

(b) Suppose \( A \) is idempotent and invertible. Show that \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Recommended reading**

Howie book, Chapter 1; you should do as many exercises as you can.