MODERN ALGEBRA I GU4041

Homework 1, due January 30: Sets and functions

1. Let $A, B, C$ be subsets of the set $X$, so that $A^c = X \setminus A$, etc.
   (a) Find shorter descriptions of the sets $(A^c \cup A^c) \cup (A \cap B) \cup (A \setminus B)$.
   (b) Prove $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ by proving that each side is contained in the other.

2. (a) Let $A$ be the set of numbers $\{2, 3, 5\}$. List all subsets of $A$. How many of the subsets contain an even number?
   (b) Let $B$ be the set of numbers $\{1, 2, \ldots, 10\}$. How many subsets of $B$ have at most 3 elements?

3. Let $\mathbb{R}$ be the set of real numbers. For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, determine whether $f$ is injective, surjective, or bijective. If $f$ is not surjective, determine its image.
   (a) $f(x) = 2x + 13$.
   (b) $f(x) = x^2$.
   (c) $f(x) = x^5$.
   (d) $f(x) = e^x - 1$.

4. Let $S$ be a set. Using the definitions carefully, show that there is exactly one function from the empty set $\emptyset$ to $S$. For which sets $S$ is this function injective? Surjective?
   Bonus question, which may help clarify the first part of Problem 4: Consider the sentence “The ocean beaches of the state of Colorado have blue sand.” Use the language of set theory, and a map of the United States, to rewrite the sentence as a clearly false statement. Using the same information, rewrite it again as a clearly true statement.

5. (a) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x \geq y\}$ and let $B = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$. Give simple descriptions of $A \cup B$ and $A \cap B$.
   (b) Let $A = \{(x, y) \in \mathbb{R}^2 \mid x > y\}$ and let $B = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$. Give simple descriptions of $A \cup B$ and $A \cap B$.

6. Let $a, b, c, d \in \mathbb{R}$. Consider the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$:
   \[ \phi(x, y) = (ax + by, cx + dy). \]
(a) Use the definitions to prove that $\phi$ is surjective if and only if, for all $e \in \mathbb{R}$, $f \in \mathbb{R}$, the linear equations

\[
L_1 : ax + by = e \\
L_2 : cx + dy = f
\]

have a solution.

(b) Show that $\phi$ is surjective if and only if $\phi$ is injective.

(c) Find numbers $a, b, c, d$, all different from 0, such that the map $\phi$ is not surjective.

7. Let $A = \{a_1, a_2, b_1, b_2\}$, and $B = \{a, b\}$.

(a) How many functions are there from $B$ to $A$? From $A$ to $B$?

(b) Write down all injective functions from $B$ to $A$, and all functions from $A$ to $B$ that are not surjective.

**Recommended reading**