SPECULATIONS ON THE MOD p REPRESENTATION THEORY OF p-ADIC GROUPS

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To Vadim Schechtman

1. Introduction

The mod p representation theory of p-adic groups began with the papers [5, 6] that treated the case of $G = GL(2, K)$, where K is a non-archimedean local field. Those papers already revealed an interesting dichotomy that continues to dominate the subject. On the one hand, if $B \subset G$ is a Borel subgroup, then any homomorphism χ from B to $\mathbb{F}^\times$, where $\mathbb{F}$ is an algebraically closed field of characteristic p, gives rise in the usual way to a principal series representation

$$I(\chi) = ind_B^G \chi = \{f : G \to \mathbb{F} \mid f(bg) = \chi(b)f(g)\}. \quad (1.1)$$

This is non-normalized induction, and a moment's thought will convince you that this is the only kind of induction possible, because the values of the modulus character equal zero in $\mathbb{F}$. One imposes the condition that $f \in I(\chi)$ is locally constant. The $I(\chi)$ are all *smooth* as this is usually understood in the representation theory of p-adic groups: every vector in $I(\chi)$ is invariant under an open compact subgroup. The irreducible constituents of $I(\chi)$ were already determined by Barthel and Livné. On the other hand, any irreducible smooth representation π of a p-adic group G is locally finite for any open compact subgroup; it follows that π is necessarily generated by a vector fixed under a given pro-p subgroup, because any finite-dimensional smooth $\mathbb{F}$-representation of a pro-p group contains a fixed vector.

In particular, letting $\mathcal{O}$ denote the ring of integers in K, $I \subset GL(2, \mathcal{O})$ an Iwahori subgroup, and $I(1) \subset I$ its maximal pro-p-subgroup, we see that any irreducible smooth π is generated by its $I(1)$-fixed vectors. In characteristic zero this would imply that π is a subquotient of a

2010 *Mathematics Subject Classification.* Primary 11F80; Secondary 11F70, 11G18, 11F67.

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement n 290766 (AAMOT). The author was partially supported by NSF grant DMS- 1404769.

principal series representation, but in characteristic p there are *supersingular* representations π that cannot be realized in this way. These are constructed by a different sort of induction: let $U = ind_{I(1)}^G 1$ be the universal module. The Hecke algebra $H(G, I(1))$ is defined to be $End_G(U)^{opp}$, so that the $I(1)$-invariants of any π becomes a left module over $H(G, I(1))$; it is canonically identified with the convolution algebra of compactly supported $\mathbb{F}$-valued functions on G that are left- and right-invariant under $I(1)$. The center $Z(G, I(1))$ of the algebra $H(G, I(1))$ is a commutative subalgebra T that is identified with the usual spherical Hecke algebra in cases of interest. For any parabolic subgroup $P \subset G$ with Levi factor M, a version of the Satake isomorphism identifies $Z(G, I(1))$ with a subalgebra of the Hecke algebra $H(M, I_M(1))$ attached to M. A quotient π of U is *supersingular* if $Z(G, I(1))$ acts on π by a character that cannot be extended to the center $Z(M, I_M(1))$ of the algebra $H(M, I_M(1))$ for any M other than G (see [1]).

Breuil gave a complete classification of the supersingular representations when $K = \mathbb{Q}_p$, and defined a Langlands parametrization of the representations of $GL(2, \mathbb{Q}_p)$ by 2-dimensional representations of the Weil-Deligne group of $\mathbb{Q}_p$. This parametrization was a landmark in the development of the p-adic local Langlands program, and was expected to set the pattern for more general groups. This was not to be, however. The analysis of parabolically induced representations continued to progress, and has culminated recently in a complete determination of irreducible constituents of such representations by Abe, Henniart, Herzig, and Vignéras [1].

However, this work, like previous results of several of the authors, reduces the classification of all irreducible smooth representations of a p-adic group G to the classification of supersingular representations of its Levi subgroups. Except when G is $GL(2, \mathbb{Q}_p)$ or $SL(2, \mathbb{Q}_p)$, the classification of supersingular representations of G is unknown and exhibits a variety of unexpected and unwelcome features. There is a functorial equivalence between irreducible 2-dimensional representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with coefficients in $\mathbb{F}$ and supersingular representations of $G = GL(2, \mathbb{Q}_p)$. When $G = GL(2, F)$ for any p-adic local field F other than $\mathbb{Q}_p$, there appear to be far more inequivalent supersingular representations of G than irreducible 2-dimensional representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$; this was shown by Breuil and Paškūnas when K is unramified and it appears to be completely general.

Two solutions are possible to this conundrum: make the set of irreducible supersingular representations of G, by imposing some additional admissibility conditions, or enhance the set of Galois representations by adding new structure. [1] The purpose of the present paper is not to opt for one solution rather than another but instead to present the situation as it might be viewed from the standpoint of geometric representation theory. This is a risky undertaking for the author but it is also risky for the reader, who should be advised that the author can claim no results in geometric representation theory and none in p-adic representation since those in his thesis, which didn't go very far.

So perhaps I should add that I would have liked to be able to offer something new as a token of my friendship with Vadik Schechtman, which began when we met in Moscow in 1989; but since I have nothing new to offer, I hope at least he will find something of interest in this survey of scattered results and barely motivated speculations.

2. Some pathologies

While reading the following list, the reader may wish to remember the following Principle, which may serve as a definition of mathematics.

Principle 2.1. *In mathematics there are no bugs, only features.*

In what follows, F is a p-adic field with integer ring $\mathcal{O}_F$ and residue field k_F and G denotes the group of F-points of a connected reductive group over F. All representations over $\mathbb{F}$ of G are assumed smooth unless otherwise indicated. The category of smooth $\mathbb{F}$-representations of G is denoted $Rep(G)$. We let $I \subset G$ be an Iwahori subgroup and $I(1) \subset I$ be its maximal pro-p subgroup. The universal module U and the Hecke algebra $H(G, I(1)) = End_G(U)$ were defined when $G = GL(2, F)$, but the same definitions make sense for any G.

2.1. **Hecke algebras.** Because every irreducible smooth representation π of G is generated by its space $\pi^{I(1)}$ of vectors under the pro-p-Iwahori subgroup $I(1)$, there is an interesting functor $\tau_{\mathbb{F}} : \pi \mapsto \pi^{I(1)}$ from $Rep(G)$ to $Mod(H(G, I(1)))$, the category of modules over the Hecke algebra $H(G, I(1))$. This and similar functors are well-known from the characteristic-zero smooth representation theory of p-adic

[1] A third possibility would be simply to define a map from the set of irreducible supersingular representations to the set of Galois parameters, and to define a "packet" to be a fiber of this map. In the local Langlands correspondence over $\mathbb{C}$, members of an L-packet are conjecturally classified by representations of the group of components of the centralizer of the parameter, and one would want to find an analogue of this structure in the mod p correspondence as well.

groups. A theorem of Borel asserts that the functor $\tau_{\mathbb{C}} : \pi \mapsto \pi^{I(1)}$ is an equivalence of categories between complex representations of G that are generated by their $I(1)$-fixed vectors and modules over the (complex) Hecke algebra of G relative to $I(1)$. However,

Theorem 2.2. [21] *Suppose $G = GL(2, E)$ where $E = F$ or $E = k((T))$ where k is a finite field of order $q = p^d$ for some d. If $\tau_{\mathbb{F}}$ is an equivalence of categories then E is a totally ramified extension of $\mathbb{Q}_p$.*

I believe it is expected that $\mathbb{Q}_p$ is in fact the only field for which the equivalence is valid, but I don't know whether or not this has been proved. The situation for groups other than $GL(2)$ (or $SL(2)$) is unclear, but it seems to be generally expected that $GL(2, \mathbb{Q}_p)$ is the exception rather than the rule. This is taken to be one reason Breuil's classification of supersingular representations has not been successfully extended to other groups.

Schneider has shown that a derived version of $\tau_{\mathbb{F}}$ does define an equivalence of derived categories; but this does not belong in the section on pathologies and it will be discussed at length below.

2.2. **Duality.** The category of smooth admissible $\mathbb{F}$-representations of G is not preserved by the natural duality. Over a field K of characteristic other than p, the functor taking a smooth admissible representation π to it smooth contragredient $\pi^{\vee}$, defined to be the subspace of $Hom(\pi, K)$ consisting of vectors invariant under an open subgroup, defines an involution of the category. Except in highly degenerate cases, when $K = \mathbb{F}$ is of characteristic p, the smooth contragredient defined as above is trivial. One is instead in a situation analogous to Pontryagin duality: if $\mathbb{F}$ is finite, for example, a smooth $\mathbb{F}$-representation π is analogous to a discrete abelian group and its algebraic dual is compact and contains no smooth vectors. Alternatively, a continuous p-adic representation τ of G on a p-adic Banach space (over some p-adic field C) is called *admissible*, in the work of Schneider and Teitelbaum, if its reduction modulo p is smooth and its linear dual is of finite type over the Iwasawa algebra of any open compact subgroup of G. The natural duality in the setting of representations of G on topological vector spaces over C thus exchanges two different categories of objects.

Kohlhaase has constructed in [18] a higher duality theory for smooth admissible $\mathbb{F}$-representations of G and shown that it has some desirable properties, especially when $G = GL(2, \mathbb{Q}_p)$. We return to this briefly below in connection with Question 4.5.

2.3. **Numerical correspondences.** The objects on the Galois side of the Langlands correspondence are continuous homomorphisms from the Weil group W_F of F to the group of $\mathbb{F}$-points of Langlands dual group ${}^LG$, up to equivalence (conjugation).[2] When $G = GL(n)$ the irreducible objects can be counted. It was proved by Ollivier [22] that the set of these objects is in bijection with the set of supersingular modules over the Hecke algebra $H(G, I(1))$. Thus there is a numerical correspondence, at least for irreducible objects. In general the set of non-trivial extensions of irreducible representations is very large (it grows with the field $\mathbb{F}$) and I don't know whether or not any kind of numerical correspondence has been attempted that takes these extensions into account; they have no counterpart in the classical theory.

Be that as it may, the category of irreducible representations π over a given $H(G, I(1))$-module M, under the functor $\pi \mapsto \pi^{I(1)}$, is in general enormous. It was first discovered by Breuil and Paškūnas in 2006 that there are far more (uncountably more) irreducible representations than Langlands parameters, except when G is $GL(2, \mathbb{Q}_p)$ or $SL(2, \mathbb{Q}_p)$ [3].

2.4. **Families.** The $\mathbb{F}$-valued characters of the center $Z(G, I(1))$ correspond to the $\mathbb{F}$-valued points of the corresponding affine scheme, and the supersingular characters form a linear subvariety of codimension equal to the rank of G^{ad}. Thus one can say that supersingular Hecke algebra modules arise by specializing families of non-supersingular modules. When $G = GL(n)$, the supersingular Hecke algebra modules are expected to correspond to irreducible n-dimensional representations of W_F (and for general G to Langlands parameters that lie in no proper parabolic subgroup of ${}^LG$). Thus a Langlands correspondence that behaves well with respect to families would seem to require generically reducible families of Galois representations with irreducible specialization.

3. Categories of Galois representations

If one wants to make the mod p local Langlands correspondence into an equivalence of categories, then homomorphisms from W_F to ${}^LG(\mathbb{F})$ are not the right object for the Galois side. Representations (or complexes of representations) of G form an additive category, whereas $Hom(W_F, {}^LG(\mathbb{F}))$ does not. The solution suggested by the geometric Langlands program is to treat $Hom(W_F, {}^LG(\mathbb{F}))$ as a stack $\mathcal{L}({}^LG)$

[2] As the referee pointed out, there is no difference between representations mod p of the Weil group and the Weil-Deligne group. If one is content to work with coefficients in the algebraic closure of a finite field, then there is no difference between representations of the Weil group and the Galois group, and most work on the p-adic Langlands correspondence has been concerned with Galois representations. However, in the geometric setting, the Weil group seems more appropriate.

and to look for an equivalence of categories between the derived category of representations of G and a derived category of (some kind of) sheaves on $\mathcal{L}({}^LG)$. For example, in the version of Arinkin and Gaitsgory [2] one considers a (derived) stack called $LocSys_{{}^LG}$ and introduces a DG-category of sheaves denoted $IndCoh_{Nilp_{glob}}(LocSys({}^LG))$ that is a full subcategory of the ind-completion of the DG-category of coherent sheaves on $LocSys({}^LG)$ adapted to account for Arthur parameters.

In the mod p theory, Emerton and Gee [12] have constructed a stack (underived) that we can call $\mathcal{L}({}^LG)(\mathbb{F})$, when $G = GL(n)$, where $\mathcal{L}$ stands for "Langlands parameter" (and also for "local system"). This is built in the first place out of families of Breuil-Kisin modules, which are (one of the) variants of Dieudonné modules that can be placed in correspondence with Galois representations with values in characteristic p coefficients. One of the salient features of this construction is that a generically reducible family of Breuil-Kisin modules can specialize to a module corresponding to an irreducible Galois representation. This fits well with the property of families of mod p representations of G already mentioned in section 2.4. Somehow the geometry of $\mathcal{L}({}^LG)(\mathbb{F})$, which is a stack over a field of characteristic p, detects properties of families in characteristic zero that are responsible for the paradoxical specialization.

The stack of Emerton and Gee seems to meet the initial requirements one might expect to be satisfied by the Galois side of the hypothetical local correspondence.[3] Let's suppose we can define a (DG) category of sheaves on this stack, denoted $?Coh(\mathcal{L}({}^LG)(\mathbb{F}))$, that has the formal properties that make it a candidate for the categorical correspondence. Taking [2] as a model, one suspects one might want the category to be compactly generated, and one might want the base to be "quasi-smooth" (i.e., a local complete intersection in the derived sense: the cohomology of the cotangent complex at each point is concentrated in degrees -1 and 0).

3.1. **Questions.** From what I understand, $\mathcal{L}({}^LG)$ is constructed (for $G = GL(n)$) by glueing special fibers of a mixed characteristic object that is something like the moduli space for semi-stable Galois representations constructed by Hartl and Hellmann as an adic space [16]. The first obvious question is

[3]The finite-type points of the stack actually correspond to n-dimensional representations of the Galois group of a deeply ramified extension of the ground field F. There are various ways to descend to the Galois group of the ground field; we will not address this question.

Question 3.1. Should $\mathcal{L}({}^LG)$ be a derived stack (constructed out of derived special fibers)?

If not, then the quasi-smoothness comes down to the local complete intersection property. I don't know whether or not the Emerton-Gee stack satisfies this property.

Question 3.2. What is the center of the category $?Coh(\mathcal{L}({}^LG)(\mathbb{F}))$?

When $G = GL(n)$ it seems clear (though I haven't seen enough of the construction) that the center should contain the ring of pseudocharacters, or more generally of determinants in the sense of Chenevier [11]. Is there more? What about more general G? A generalization of pseudocharacters has been defined by Vincent Lafforgue in his construction of Langlands parameters attached to automorphic representations over function fields ([19], Proposition 11.7). His method only works with coefficients in an algebraically closed field of characteristic zero. Thus one is led to the following questions:

Question 3.3. Are the data used in Lafforgue's construction (there denoted Ξ_n) represented by a noetherian ring? Or by a derived stack? Does this ring (or derived stack) have a natural model over $Spec(\mathbb{Z})$?

Let Γ be a profinite group. (In [19] $\Gamma = Gal(\bar{F}/F)$ where F is a global field; I don't know whether or not Lafforgue has developed a version for the Weil-Deligne group.) For each $n > 0$, $\Xi_n(\mathcal{B})$ is defined in ([19] 11.3) to be an algebra homomorphism from $\mathcal{O}({}^LG^n//{}^LG)$ to $C(\Gamma^n, \mathcal{B})$; here ${}^LG^n//{}^LG$ is the geometric invariant theory quotient of ${}^LG^n$ under simultaneous conjugation by ${}^LG$, viewed as a scheme over a topological base ring E (an ℓ-adic field in [19]), $\mathcal{B}$ is a topological E-algebra, and $C(\bullet, \mathcal{B})$ denotes continuous functions. The $\mathcal{B}$ is not in Lafforgue's notation, but if we write it this way we see it is the set of $\mathcal{B}$-valued points of the "space" M_n of continuous maps (in some sense) from Γ^n to ${}^LG^n//{}^LG$. As n varies, the Ξ_n satisfy certain recurrence relations, that appear to make the collection $\Xi_\bullet$ into the set of $\mathcal{B}$-valued points of the simplicial "space" $M_\bullet$ of continuous maps from $B(\Gamma)$ to $B({}^LG)//{}^LG$.

This construction may or may not have a rigorous meaning. When E is replaced by the finite field $\mathbb{F}$ with the discrete topology, the continuity may be moot.

Question 3.4. What is the relation of the center of the category $?Coh(\mathcal{L}({}^LG)(\mathbb{F}))$ and the simplicial "space" $M_\bullet$ described above?

Lafforgue explains how his construction is equivalent to the construction of pseudocharacters for $GL(n)$ over a field of characteristic zero.

The same should be true when $p > n$; for small p Chenevier's determinants [11] provide a substitute. I don't know whether or not anyone has attempted to to extend Chenevier's construction to groups other than $GL(n)$.

Remark 3.5. The pseudocharacter of a Galois representation plays a small but essential role in the Taylor-Wiles method of proving modularity of p-adic representations of Galois groups of number fields. The Taylor-Wiles method, and its various extensions, starts with a surjective homomorphism from the (p-adic) deformation ring R of an absolutely irreducible representation of the Galois group over $\mathbb{F}$ to a related p-adic Hecke algebra T, and uses arguments from Galois cohomology and automorphic forms to show that this is in fact an isomorphism. The existence of the map depends on a theorem of Carayol that guarantees that the deformation of the Galois representation obtained using automorphic forms can be realized with coefficients in T, provided its trace – in other words, its pseudocharacter – lies in T. A generalized pseudocharacter seems necessary in order to extend the Taylor-Wiles method to a general group ${}^LG$, without dependence on an embedding in $GL(n)$.

4. Derived Hecke algebras

We assume in what follows that the pro-p-Iwahori subgroup $I(1) \subset G$ is torsion-free. This is true generically (when p is large relative to the root system of G and the ramification degree of $F/\mathbb{Q}$ is small).

Recall that the (pro-p-Iwahori) Hecke algebra $H(G, I(1))$ was defined to be $End_G(U)$, where U is the universal smooth mod p representation of G. I call it "universal" because it maps surjectively to any irreducible $\mathbb{F}$-representation of G, for the reasons explained above. Let

$$H^{\bullet}(G, I(1)) = RHom^{\bullet}(U, U)^{opp}$$

which is well defined in the homotopy category of DG-algebras.

We have explained (in Theorem 2.2 and the subsequent discussion) that the functor

$$Rep(G) \to Mod(H(G, I(1)); \pi \mapsto \pi^{I(1)} \quad (4.1)$$

was shown in [21] to be an equivalence of categories for $GL(2, F)$ if $F = \mathbb{Q}_p$ but not if F is of characteristic p nor if the residue field of F strictly contains $\mathbb{F}_p$; it is not known what happens for more general G. However, Schneider has shown in [24] that, provided $I(1)$ is torsion-free, the derived version of the functor (4.1) defines an equivalence of triangulated categories between the (unbounded) derived categories

$$H : D(Rep(G)) \to D(Mod(H^{\bullet}(G, I(1))). \tag{4.2}$$

The main step in Schneider's proof is to show that the functor H is conservative, which is non-trivial; the rest of the proof is along the familiar lines of the Barr-Beck theorem on adjoint functors.

4.1. **Questions.** Very little is known about the structure of the DG algebra $H^{\bullet}(G, I(1))$. For example, the center of the (underived) Hecke algebra $H(G, I(1))$ has been determined (in several ways: see [29], [22]). As far as I know, no one knows the answer to the following question:

Question 4.3. What is the center of the DG algebra $H^{\bullet}(G, I(1))$? What is its relation, if any, to the center of $H(G, I(1))$?

Here I have to pause to mention that there is more than one possible notion of center of a DG algebra $A^{\bullet}$, and the relations between these are not clear to me. Experts tell me that the preferred notion is given by the Hochschild cohomology of $A^{\bullet}$, and that this is already a derived object. A standard reference is [7], which is written in the generality of monoidal ∞-categories, perhaps more generality than is strictly necessary for such a concrete object as $H^{\bullet}(G, I(1))$. Even if the Hochschild cohomology is deemed to be defined by the standard bar complex, I wouldn't know how to begin to compute it explicitly. Given that the center of $H(G, I(1))$ has a simple presentation, this is somewhat surprising. In any case, there is in general a canonical map from $HH^0(A^{\bullet})$ to the center of $H^0(A^{\bullet})$. Thus we can add a pendant to Question 4.3:

Question 4.4. Is the canonical map from $HH^0(H^{\bullet}(G, I(1)))$ to the center of (the underived Hecke algebra) $H(G, I(1))$ surjective?

Schneider's canonical construction of the DG algebra $H^{\bullet}(G, I(1))$ is unbounded in both directions, so the sense in which the center of $H(G, I(1))$ acts on an $H^{\bullet}(G, I(1))$-module is not immediately clear. Abouzaid has explained to me how to construct a canonical model of $H^{\bullet}(G, I(1))$ as a DG algebra concentrated in non-negative degrees. Schneider's original paper [24] shows that $H^i(G, I(1))$ vanishes for i not in $[0, \dim G]$; this definitely fails if $I(1)$ has torsion. The higher derived Hecke algebras $H^i(G, I(1))$ are modules over $H(G, I(1)) = H^0(G, I(1))$. It is not hard to compute $H^i(G, I(1))$ as an $\mathbb{F}$-vector space for every i, but the algebra structure is practically unknown. Schneider has computed $H^{dimG}(G, I(1))$ explicitly ([24] Proposition 6) but even there the module structure is unclear. The only non-trivial results seem to be due to Ollivier and Schneider; they have shown, for instance, that, when $G = GL(2, F)$, $H^1(G, I(1))$ contains a non-zero torsion submodule over $H(G, I(1))$ unless $F = \mathbb{Q}_p$, and that this torsion submodule consists of supersingular modules; this is related to the failure of the functor (4.1) to be an equivalence of categories.

An admissible $\pi \in Rep(G)$ defines a module of finite type $H(\pi)$ over $H^\bullet(G, I(1))$, via the functor H of (4.2); and this remains true when π is replaced by an admissible complex $\pi^\bullet$, whose definition we leave to the reader. Now the linear dual of a left $H^\bullet(G, I(1))$-module of finite type is a right $H^\bullet(G, I(1))$-module of finite type. However, the anti-involution $g \mapsto g^{-1}$ of G interchanges right and left $H^\bullet(G, I(1))$-modules, so there is a duality involution δ on (the bounded derived category) $D^b(Mod(H^\bullet(G, I(1)))$.

It's not at all clear (to me) whether or not the functor H of (4.2) restricts to an equivalence of bounded derived categories, so we will write $D^b_H(Rep(G))$ for the full subcategory of $D(Rep(G))$ that corresponds to $D^b(Mod(H^\bullet(G, I(1)))$ under H.

Question 4.5. What is the duality involution on $D^b_H(Rep(G))$ that corresponds to δ on $D^b(Mod(H^\bullet(G, I(1)))$? What is its relation, if any, to the higher duality theory constructed by Kohlhaase in [18]?

In this connection, it should be noted that Kohlhaase constructs a sequence of smooth duality functors S^i by sending an admissible representation π to the Ext^i of the Pontryagin dual of π with a dualizing module. As in earlier work of Schneider-Teitelbaum and Venjakob, the category of admissible representations has a filtration and Kohlhaase defines the notion of *Cohen-Macaulay* objects to be those π for which S^i vanishes outside a single dimension $d(\pi)$. He verifies that supercuspidal representations and the Steinberg representation of $GL(2, \mathbb{Q}_p)$ are Cohen-Macaulay in this sense with $d(\pi) = 1$. In general, $d(\pi)$ can vary for Cohen-Macaulay representations of a given group G, which lends credence to the suggestion by Ben-Zvi that $D^b_H(Rep(G))$ has a non-obvious t-structure adapted to a hypothetical congruence to objects on the Galois side.

When $I(1)$ is no longer assumed to be torsion-free, one can replace $I(1)$ by a torsion-free subgroup of finite index, and the theory goes through, but the meaning of the functor H is no longer clear. It may or may not be relevant that the theory of pseudocharacters of $GL(n)$ also fails to work when $p \leq n$.

4.2. **Correspondences.** As noted in Section 2.3, there are far too many irreducible supersingular objects in $Rep(G)$ (except when $G = GL(2, \mathbb{Q}_p)$ or $SL(2, \mathbb{Q}_p)$) to match the available stock of Galois objects (skyscraper sheaves on $\mathcal{L}({}^LG)$). This is one reason the categorical approach seems promising. However, the theory of mod p cohomology of locally symmetric spaces furnishes a large collection of admissible representations of G. In many cases this cohomology can be naturally associated to a Galois representation; more generally, completed cohomology yields modules with commuting actions of G and $Gal(\bar{F}/F)$. This global construction provides a candidate for a local correspondence; this has been studied for $GL(n)$ in many cases in the paper [10].

Let E be a number field with a p-adic completion isomorphic to F. Suppose for simplicity that G is split over F. Let ρ be a (modular) representation of $Gal(\overline{\mathbb{Q}}/E)$ with values in ${}^LG(\mathbb{F})$. The (still mostly conjectural) generalizations of Serre's conjecture assign to ρ a set $S(\rho)$ of irreducible modular representations of $G(k_F)$, the so-called "Serre weights" of ρ; they are identified by their highest weights. (Actually, as far as I know these have only been defined when $G = GL(n)$, and for $U(3)$; the case of $GL(2)$ was first studied by Buzzard, Diamond, and Jarvis, and the case of general $GL(n)$ by Herzig, extending earlier work of Ash, Doud, and Pollack.) If $\pi \in Rep(G)$, we can define the socle of π to be the maximal semisimple $G(\mathcal{O}_F)$-subrepresentation of π. Let $G = GL(2, F)$, with $F \neq \mathbb{Q}_p$. Given ρ, the paper [3] then gives a recipe (in section 11) for the socle of π in terms of the "Serre weights" of ρ, following [9]; the authors of [3] observe that there are irreducible π whose socles can't possibly correspond to Galois representations ρ.

Question 4.6. Can this socle condition be interpreted cohomologically?

I would guess not. In particular, there doesn't seem to be a reasonable definition of subcategory of $D(Rep(G))$ that distinguishes those objects that belong in a correspondence with objects in $?Coh(\mathcal{L}({}^LG)(\mathbb{F}))$ from those that don't. Nor is it clear whether the socle condition defines a meaningful restriction on $D(Rep(G))$ at all. Nevertheless, the possibility of such restrictions should be taken into consideration when reacting to the following question:

Question 4.7. Is there an equivalence of (derived or DG or ∞-) categories between $?Coh(\mathcal{L}({}^LG)(\mathbb{F}))$ and $D(Mod(H^\bullet(G, I(1))))$? Or between naturally defined subcategories of the two sides that include everything that arises in the cohomology of locally symmetric spaces?

A first step in evaluating whether or not this is reasonable would be to compare the answers to Questions 3.2 and 4.3. In this connection, Corollary 8.11 of [23], which treats the case of $GL(2, \mathbb{Q}_p)$, is extremely suggestive. I note that Helm has announced in [17] a conjectural answer to the analogous question in the case of ℓ-adic representations of $GL(n,F)$, with F a p-adic field, $p \neq \ell$.

5. Geometric correspondences

Three proposals to define functorial correspondences between mod p representations of G and Galois representations have been discussed in public since the beginning of 2014. The most complete was announced in Peter Scholze's closing lecture at the MSRI Hot Topics workshop on perfectoid spaces on February 21. A set of notes is available on the MSRI website [25]. The construction applies only to $GL(n,F)$ and is based on the Gross-Hopkins period map

$$\pi_{GH} : \mathcal{M}_{LT,\infty} \to \mathbb{P}^{n-1}_{\breve{F}}. \tag{5.1}$$

Here $\breve{F}$ is the p-adic completion of the compositum of F with the fraction field of the Witt vectors of the algebraic closure of the residue field of F and $\mathcal{M}_{LT,\infty}$ is the perfectoid Lubin-Tate moduli space over $\breve{F}$. This space has a continuous action of $GL(n,F) \times D^\times$, where D is the central division algebra over F with invariant $\frac{1}{n}$.

The map π_{GH} is to be understood as a map of adic spaces, and makes $\mathcal{M}_{LT,\infty}$ an étale $GL(n,F)$-torsor over $\mathbb{P}^{n-1}_{\breve{F}}$. Thus any admissible $\mathbb{F}[GL(n,F)]$-module π gives rise to a (pro-étale) sheaf $\mathcal{F}_\pi$ over $\mathbb{P}^{n-1}_{\breve{F}}$. Fix a complete algebraically closed extension $C/\breve{F}$. The cohomology groups $H^i(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$ carry a continuous action of $D^\times \times W_F$, where W_F is the Weil group of F. Scholze has announced the following result, whose proof is sketched in [25]:

Theorem 5.2 (Scholze)**.** *The cohomology groups $H^i(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$ are independent of C and vanish for $i > 2(n-1)$. As a representation of $D^\times$, each $H^i(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$ is admissible.*

The proof is a stunning application of Scholze's perfectoid techniques. One is tempted to conjecture that $\oplus_i H^i(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$ realizes at least a part of a (graded) mod p local Langlands correspondence for the group $GL(n,F)$, together with a mod p Jacquet-Langlands correspondence. Scholze does not go so far as to state a conjecture but he does provide some evidence in the form of compatibility with the global correspondence on (at least a part of) the mod p cohomology of appropriate Shimura varieties (see Proposition 11 of [25]).

It may be possible to extend the methods of [25] to groups G other than $GL(n,F)$, but only in the setting of the Rapoport-Zink spaces attached to minuscule weights. One of the purposes of the course [26] was to remove this restriction by enriching the category of p-adic spaces on which G acts. The introduction to [26] expresses the "hope" that the moduli spaces of *local shtukas* constructed there can play the role of the generalized Rapoport-Zink spaces whose conjectural properties (and existence) have been described by Rapoport and Viehmann. In particular, the cohomology of these spaces would also provide candidates for a hypothetical mod p (and p-adic) local Langlands correspondence; however, in informal remarks Scholze has mentioned that he is focusing on ℓ-adic cohomology, at least for the time being.

A third approach will be suggested by Fargues in reference [9] in the bibliography of [14]. This paper has not yet been made public, but its contents were presented during Fargues' talk at the MSRI workshop in 2014; the link to the video of his talk on the MSRI website seems to be missing. Fargues presented a conjectural cohomological construction of the ℓ-adic local Langlands correspondence for a group $G/\mathbb{Q}_p$, for $\ell \neq p$. However, discussions between Fargues and Scholze appear to have convinced one or both of them that a version of this conjecture should be valid for $\ell = p$ as well.

The relation, if any, between the geometric constructions of Scholze and Fargues and the hypothetical categorical correspondence described in Section 4.2 is by no means clear. The compatibility with global correspondences suggests that representations that fail the socle condition of [3] (see 4.6) should not contribute to the correspondences just discussed. x

6. Characters

Let D be the central division algebra over F introduced in the previous section, and let $G = D^\times/F^\times$. This is a compact profinite p-adic analytic group but it is also the group of F-rational points of an algebraic group over F. An admissible representation (σ, V) of G over $\mathbb{F}$ is the direct limit of the finite-dimensional $\mathbb{F}[G_n]$-modules V^{U_n}, where U_n runs through a nested sequence of open normal subgroups of G, $G_n = G/U_n$, and $\cap_n U_n = \{1\}$.

As an approximation to the center of the category of admissible complexes of $\mathbb{F}[G]$-modules (however this admissibility is defined), one might consider $HH^*_{sm}(\mathbb{F}[G]) = \varprojlim_n HH^*(\mathbb{F}[G_n]))$, with the natural morphisms from $HH^*(\mathbb{F}[G_n]) \to HH^*(\mathbb{F}[G_{n+1}])$ defined relative to the projection maps $G_{n+1} \to G_n$. The advantage of working with finite groups is that the Hochschild cohomology groups of their group algebras can be computed in terms of conjugacy classes. Following the discussion in section 7.4 of [20], which treats Hochschild homology, one can write $HH^*(\mathbb{F}[G_n])$ as a direct sum over conjugacy classes $[\gamma_n]$ of elements $\gamma_n \in G_n$ of the cohomology groups $H^*(G_{n,\gamma_n}, \mathbb{F})$, where G_{n,γ_n} is the centralizer of γ_n (see [20], Theorem 7.4.6; the isomorphism can even be lifted to the chain level). If we restrict attention to sequences of $(\gamma_n \in G_n, n \geq 0)$ where each γ_n is the reduction mod U_n of a (necessarily elliptic) regular element $\gamma \in G$, with centralizer G_γ, then the corresponding piece of $HH^*_{sm}(\mathbb{F}[G])$ looks in the limit like the exterior algebra on $\Omega^1(G_\gamma) = Hom_{\mathcal{O}_F}(Lie(G_\gamma), \mathbb{F})$; here G and its subgroups G_γ are given the natural structure of schemes over the integer ring $\mathcal{O}_F$. Since the choice of $\gamma_n \in [\gamma_n]$ is not canonical, one might replace $\Omega^1(G_\gamma)$ by the conormal bundle to the conjugacy class of $\gamma \in G$.

If I understand the point of [28] and [7] correctly, the Chern character of an admissible representation (σ, V) of G would then define a map from $HH^*_{sm}(\mathbb{F}[G])$ to $\mathbb{F}$, which could be evaluated on $\bigwedge^\bullet \Omega^1(G_\gamma)$ for each conjugacy class $[\gamma]$. This is of course wildly speculative, not least because I have no idea how one would go about computing such a pairing between representations and exterior differentials on Lie algebras of centralizers. What I find intriguing, however, is that the regular conjugacy classes can be transferred to inner forms of G, notably to $PGL(n, F)$. The transferred conjugacy classes have representatives in a maximal compact open subgroup $K \subset PGL(n, F)$, whose centralizers bear the same relation to $HH^*_{sm}(\mathbb{F}[K])$ as the centralizers of the original conjugacy classes bear to $HH^*_{sm}(\mathbb{F}[G])$. This makes it possible, at least in principle, to compare Chern characters of admissible representations of G and its inner forms. This would provide a natural test, as a first approximation, of the naturality of the mod p Jacquet-Langlands correspondence constructed by Scholze, see Theorem 5.2. Since the characters fit naturally in the framework of the categorical theory of traces developed in [8], it may be possible to use a Lefschetz formalism to carry out this comparison, as in [13, 27]; see also Chapter 9 of [15].

Acknowledgements

I am indebted to the referee for an indulgent and thoughtful reading and for a number of illuminating remarks. I would also like to thank Mohammed Abouzaid, David Ben-Zvi, Christophe Breuil, Matthew Emerton, Carl Wang Erickson, Laurent Fargues, Edward Frenkel, Dennis Gaitsgory, Toby Gee, Alain Genestier, Vincent Lafforgue, Rachel Ollivier, Vytas Paškūnas, Sam Raskin, Nick Rozenblyum, Peter Scholze, Peter Schneider, and Gabriele Vezzosi for many helpful discussions. They are in no way responsible for any of the author's blunders, which

are likely to be considerably more numerous in this article than in others of the same length.

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