Eisenstein cohomology and special values of *L*-functions

Michael Harris

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Outline

Automorphic vs. motivic L-functions: motivic L-functions

Motivic L-functions, attached to representations of Galois groups of number fields, have [conjecturally] arithmetically meaningful special values at integer points.

Example: $\zeta(2) = \frac{\pi^2}{6}$. The π^2 reflects the relation of $\zeta(2)$ to the square of the cyclotomic character, the denominator reflects deeper properties of cyclotomic fields.

Deligne's conjecture: certain special values, called *critical*, are related to determinants of integrals of arithmetic differential forms (de Rham cohomology of an algebraic variety whose ℓ-adic cohomology contains the Galois representation) over topological cycles on the same variety.

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Deligne's conjecture

Let M be a motive of rank n over $\mathbb{Q}$, which can be identified with a compatible family of ℓ-adic Galois representations $\rho_{\ell,M}$ of rank n. Then we define the L-function $L(s,M) = \prod_p L_p(s,M)$ where for almost all p,

$$L_p(s,M) = [\det(1 - \rho_{\ell,M}(Frob_p)T)^{-1}]_{T=q^{-s}}$$

Let $s_0 \in \mathbb{Z}$ be a critical value of $L(s,M)$. (A crude version of) Deligne's conjecture:

$$L(s_0,M) \sim c^+(s_0,M)$$

where $c^+(s_0,M)$ is a certain determinant of periods of differential forms on M twisted by $\mathbb{Q}(s_0)$ [sic!] and $\sim$ means "up to $\overline{\mathbb{Q}}$-multiples.

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Automorphic vs. motivic L-functions: automorphic L-functions

Automorphic L-functions generally have no obvious connection to arithmetic, but their special values are often expressed as integrals of differential forms over locally symmetric varieties. Example: Rankin-Selberg L-function of $GL(n) \times GL(n-1)$.

Goal (in light of Deligne's conjectures): relate such integrals to arithmetic integrals on Shimura varieties. This is done by proving *period relations* when the same L-function has different integral representations (on different groups).

The Rankin-Selberg L-function of $GL(n) \times GL(1)$, in some cases, is the standard L-function of a unitary group with integral representation on a Shimura variety (hermitian locally symmetric space).

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Description of main results

Joint work with Grobner, with Grobner-Lapid, and more recent results in the thesis of Lin Jie.

Let $\mathcal{K}$ be a CM field, i.e. a totally imaginary quadratic extension of a totally real number field F. (May assume $F = \mathbb{Q}$, $\mathcal{K}$ imaginary quadratic.) An automorphic representation Π of $GL(n)_{\mathcal{K}}$ is *conjugate-dual* if $\Pi^{\vee} = \Pi^{c}$; $c \in Gal(\mathcal{K}/F)$ is complex conjugation.

1. We relate critical values of $L(s, \Pi \times \Pi')$, Π on $GL(n)_{\mathcal{K}}$, Π' on $GL(n-1)_{\mathcal{K}}$, to Whittaker periods.
2. We relate Whittaker periods of (cuspidal) Π to periods of holomorphic forms on Shimura varieties.
3. When $F \neq \mathbb{Q}$, we [i.e., Lin Jie] express these periods as products over the real places of F of periods on simpler Shimura varieties.

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The JPSS zeta integral

Let $G = GL(n)$, $G' = GL(n-1)$. Let $\Pi \times \Pi'$ be an automorphic representation of $G \times G'$ over $\mathcal{K}$, Π cuspidal,

$$\iota : G' \hookrightarrow G, \ \iota(g') = diag(g', 1)$$

The JPSS zeta integral for $G \times G'$ over $\mathcal{K}$ is

$$Z(s, \phi, \phi') = \int_{G'(\mathcal{K})\backslash G'(\mathbf{A})} \phi(\iota(g'))\phi'(g')||det(g')||^s dg'.$$

When Π and Π' are cohomological representations, and ϕ, ϕ' cohomological vectors, $Z(s, \phi, \phi')$ can be interpreted as a cup product on the locally symmetric space for $G \times G'$.

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The cup product

If Π is cuspidal and $H^i(\mathfrak{gl}(n), U(n); \Pi_\infty \otimes W) \neq 0$ for some finite dimensional representation W then the lowest degree i is $b_n = \frac{n(n-1)}{2}$ and $\dim H^{b_n} = 1$.

Now $b_n + b_{n-1} = (n-1)^2 = \dim G'(\mathbb{R})/U(n-1) = \dim {}_K\tilde{S}_{n-1}$, where

$$ {}_K\tilde{S}_{n-1} = G'(\mathcal{K})\backslash G'(\mathbf{A})/U(n-1) \times K, \; K \subset G'(\mathbf{A}_f). $$

Thus the expression $\phi(\iota(g'))\phi'(g')$ is a top degree differential $\omega_\phi \cup \omega_{\phi'}$ on ${}_K\tilde{S}_{n-1}$ for appropriate choice of ϕ, ϕ' and the JPSS integral is its image in top-degree compactly-supported cohomology.

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If Π is cuspidal and $H^i(\mathfrak{gl}(n), U(n); \Pi_\infty \otimes W) \neq 0$ for some finite dimensional representation W then the lowest degree i is $b_n = \frac{n(n-1)}{2}$ and $\dim H^{b_n} = 1$.

Now $b_n + b_{n-1} = (n-1)^2 = \dim G'(\mathbb{R})/U(n-1) = \dim {}_K\tilde{S}_{n-1}$, where

$${}_K\tilde{S}_{n-1} = G'(\mathcal{K})\backslash G'(\mathbf{A})/U(n-1) \times K, \; K \subset G'(\mathbf{A}_f).$$

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The fine print

Assume

Hypotheses (Good Position Hypotheses)

1. Π'_∞ *tempered,* $H^{b_{n-1}}(\mathfrak{gl}(n-1), U(n-1); \Pi'_\infty \otimes W') \neq 0$.
2. $Hom_{\mathbb{C}}(W \otimes W', \mathbb{C}) \neq 0$ *(in which case the space is* 1*-dimensional.*

Otherwise the cup product vanishes.
The JPSS integral has an Euler product: if S is the (finite) set of archimedean and ramified primes, then

$$Z(s, \phi, \phi') = \prod_v Z_v(s, \phi_v, \phi'_v) = \prod_{v \notin S} Z_v(s, \phi_v, \phi'_v) \times L^S(s, \Pi_v, \Pi'_v).$$

Can ignore non-archimedean places in S; we'll see about the archimedean factors $Z_v(s, \phi_v, \phi'_v)$ later.

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Rationality

$\Rightarrow$ relation between $L(s_0, \Pi, \Pi')$ and $Tr[\omega_\phi \cup \omega_{\phi'}]$ *for* s_0 *critical.*

ϕ, ϕ' *cohomologically rational* :

$$\Rightarrow Tr[\omega_\phi \cup \omega_{\phi'}] \in \overline{\mathbb{Q}}.$$

ϕ, ϕ' *Whittaker rational* (Fourier coefficients in $\overline{\mathbb{Q}}$) :

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Factors of proportionality in $\mathbb{C}^\times$, well-defined up to $\overline{\mathbb{Q}}$-multiples, relate these two $\overline{\mathbb{Q}}$-structures: call them $p(\Pi)$ and $p(\Pi')$. Then

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Archimedean factors

We assume

Hypotheses (Polarization Hypothesis)

$\Pi^\vee \simeq \Pi^c$, $\Pi'^{,\vee} \simeq \Pi'^{,c}$, *where ^{c} is the action of Galois conjugation.*

B.-Y. Sun has shown that the complex number $p_\infty \neq 0$ under the Good Position Hypotheses. Lin Jie has used period relations (see below) to identify p_∞ as a power of π, up to algebraic factors.

But what is the relation of $p(\Pi)$ to algebraic geometry, i.e. to $c^+(M)$, where $M = M(\Pi)$ is a motive attached to Π?

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Shahidi's formula

Suppose $\Pi' = \Pi_1 \times \cdots \times \Pi_r$, Π_i on $GL(n_i)$, $\sum_i n_i = n - 1$.

Theorem (Shahidi, essentially)

$$p(\Pi') \sim c_\infty \cdot \prod_{i=1}^{r} p(\Pi_r) \times \prod_{1 \leq i < j \leq r} L(1, \Pi_i \otimes \Pi_j^\vee).$$

Also, if $n_i = 1$, then $p(\Pi_i) \sim 1$.

In Grobner-H. this is applied when each $n_i = 1$ and the $\Pi_i = \chi_i$ are algebraic Hecke characters. Then

$$L(1, \Pi_i \otimes \Pi_j^\vee) = L(1, \chi_i/\chi_j) = p(\chi_i/\chi_j)$$

is a period of a CM abelian variety (Damarell, Shimura, Blasius). Thus $p(\Pi')$ is a product of CM periods (known quantities).

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Relation to unitary Shimura varieties, 1

On the other hand, the polarization hypothesis implies

1. Each of Π, Π' descends to holomorphic representations π_a, π'_b of unitary groups $U(a, n-a)$, $U(b, n-1-b)$ for all a, b [some local hypotheses may be necessary]
2. The Deligne period $c^+(M(\Pi) \otimes M(\Pi'))$ has a simplified expression [see below]

Concretely, this means that for each a there is a holomorphic modular form $f_a \in \pi_a$ on $U(a, n-a)$ with arithmetic normalization such that $L(s, \pi_a) = L(s, \Pi)$ where $L(s, \pi_a)$ is the *standard L-function* (PS-Rallis).

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Suppose $L(s, \Pi \otimes \chi_i)$ has a non-vanishing critical value for each i. (automatic if Π is sufficiently regular). Then

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Proposition (MH,Lin)

Let Π_1 *cuspidal for* $GL(n_1)$, Π_2 *cuspidal for* $GL(n_2)$, $n_2 < n_1$. *Assume* Π_1 *and* Π_2 *satisfy the Polarization Hypothesis and let* s_0 *be critical for* $L(s, M(\Pi_1) \otimes M(\Pi_2))$.

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with same $\mu(n_1, n_2), s(\Pi_\infty, \Pi'_\infty, a), s'(\Pi'_\infty, \Pi_\infty, b)$.

2. Without Good Position Hypothesis, she proves this for $s_0 = 1$ using Shahidi's formula, for very regular Π_1, Π_2.
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Shahidi's formula computes $p(\Pi')$ for any tempered Eisenstein class, with $\sum m_i = n-1$:

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