# ALGEBRAIC NUMBER THEORY W4043

## 1. Homework, week 2, due September 19

1. Denote by $M(n, \mathbb{Z})$ the ring of $n \times n$ matrices with coefficients in $\mathbb{Z}$. Let $\alpha \in M(n, \mathbb{Z})$. Show that every element of the subring $\mathbb{Z}[\alpha] \subset M(n, \mathbb{Z})$ is integral over $\mathbb{Z}$.

2. Let $R$ be a Dedekind domain with a finite number of distinct prime ideals. Show that it is a principal ideal domain. (Hint: Let $\mathfrak{p}$ be a prime ideal of $R$. Use the Chinese remainder theorem to find an element $x \in \mathfrak{p}$ such that $(x) = \mathfrak{p}$.)

3. Let $R$ be an integral domain with fraction field $K$. A *multiplicative subset* $S \subset R$ is a subset such that,

- $1 \in S$, $0 \notin S$;
- If $s, s' \in S$ then $ss' \in S$.

The *localization* $S^{-1}R$ is the subset of $K$ consisting of elements $\frac{r}{s}$ with $r \in R$ and $s \in S$. (Alternatively, it is the set of equivalence classes of pairs $(r, s)$, with $r \in R$ and $s \in S$, with $(r, s)$ equivalent to $(r', s')$ if and only if $rs' = r's$).

(Localization is also defined for general commutative rings, but the definition is more elaborate.) After convincing yourself that $S^{-1}R$ is a ring, show that

(a) If $S$ is the set of non-zero elements of $R$, then $S^{-1}R = K$;

(b) If $R$ is a Dedekind domain, then so is $S^{-1}R$ for any multiplicative subset $S \subset R$.

(c) If $I \subset R$ is an ideal, let $S^{-1}I \subset S^{-1}R$ be the ideal of $S^{-1}R$ generated by $I$. Show that the map

$$I \mapsto S^{-1}I$$

is a surjection from the set of ideals of $R$ to the set of ideals of $S^{-1}R$. Use the proof to construct a bijection between the set of prime ideals of $S^{-1}R$ and the subset of prime ideals $\mathfrak{p} \subset R$ such that $\mathfrak{p} \cap S = \emptyset$.

(d) Let $R$ be a Dedekind domain, $\mathfrak{p} \subset R$ be a prime ideal, let $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$, and define $R_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}R$. Show that $R_{\mathfrak{p}}$ is a *discrete valuation ring*, i.e. a Dedekind domain with a unique non-zero prime ideal. In particular, show (using problem 2) that every non-zero element $a \in R_{\mathfrak{p}}$ has a unique factorization of the form $a = uc^b$, where $c$ is a generator of the unique non-zero prime ideal of $R_{\mathfrak{p}}$, $b$ is a non-negative integer, and $u$ is an invertible element of $R_{\mathfrak{p}}$.