# ALGEBRAIC NUMBER THEORY W4043

## Homework, week 7, due November 1

1. (a) Let $q(X,Y) = aX^2 + bXY + cY^2$ be a positive-definite binary quadratic form with integer coefficients. Assume it has discriminant $\Delta = -7$ and is *reduced*. Recall that a reduced quadratic form has the property that $a \leq \sqrt{|\Delta|/3} \approx 1.53$. Give the possible values for $(a,b,c)$.

(b) Use the result of (a) to determine the class number of $K = \mathbb{Q}(\sqrt{-7})$.

(c) For each $q$ as in (a), determine the set of primes $p$ represented by $q$. What is their relation to the set of primes that split in $K$?

## Dirichlet characters

Let $n$ be a positive integer. A *Dirichlet character* modulo $n$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

(1) $\chi(ab) = \chi(a)\chi(b)$.
(2) $\chi(a)$ depends only on the residue class of $a$ modulo $n$.
(3) $\chi(a) = 0$ if and only if $a$ and $n$ have a non-trivial common factor.

It follows that a Dirichlet character modulo $n$ can also be considered a function $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$.

Let $X(n)$ denote the set of distinct Dirichlet characters modulo $n$. We consider $X(p)$ when $p$ is prime and show it forms a cyclic group with identity element $\chi_0$ defined by $\chi_0(a) = 1$ if $(a,p) = 1$, $\chi_0(a) = 0$ if $p \mid a$.

2. Show that for any $\chi \in X(p)$, $\chi(1) = 1$, and $\chi(a)$ is a $(p-1)$st root of 1 for all $a \in (\mathbb{Z}/p\mathbb{Z})^\times$.

3. For all $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, show that $\chi(a^{-1}) = \bar{\chi}(a)$ where $\bar{\chi}$ is the complex conjugate function.

4. Show that $\sum_{a \in \mathbb{Z}/p\mathbb{Z}} \chi(a) = 0$ if $\chi \neq \chi_0$.

5. Show that the Legendre symbol $a \mapsto \left(\frac{a}{p}\right)$ for $(a,p) = 1$, extended to take the value 0 at integers divisible by $p$, defines a Dirichlet character modulo $p$ that is different from $\chi_0$.