ALGEBRAIC NUMBER THEORY W4043

1. Homework, week 6, due October 18

If K is a number field, the letters r_1 and r_2 designate respectively the number of real embeddings and pairs of complex conjugate embeddings of K.

1. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $X^7 - 1$.

(a) List the intermediate fields K_i between K and $\mathbb{Q}$ and for each i, use Galois theory to find $\alpha_i \in K$ such that $K_i = \mathbb{Q}(\alpha_i)$.

(b) Show that there is a unique subfield $K' \subset K$ such that $[K' : \mathbb{Q}]$ is quadratic. Determine the set of primes $p \in \mathbb{Q}$ that ramify in K', and use this to write $K' = \mathbb{Q}(\sqrt{d})$ for some integer d. What are r_1 and r_2 for K'?

(c) For any $n > 1$, show that $\phi(n)$ is divisible by 2. Let K_n be the splitting field over $\mathbb{Q}$ of the polynomial $X^n - 1$. List all the subfields $L \subset K_n$ such that $[L : \mathbb{Q}] = 2$. If $n = p_1 \cdot p_2$ is the product of two odd primes, determine r_1 and r_2 for each such L.

2. The function $D(x_1, x_2, \ldots, x_n)$ in Hindry's exercise 6.15 of Hindry's book, p. 120, is called the *discriminant* of the basis $(x_1, \ldots, x_n)$. Compute discriminants of several bases of of the ring of integers in $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer.

3. In the notation of Hindry's exercise, we let p be a prime number, r a positive integer, and let $F(X) = (X^{p^r} - 1)/(X^{p^{r-1}} - 1)$, a polynomial of degree $\phi(p^r)$ where ϕ denotes the Euler function. Let $K = \mathbb{Q}(\zeta)$ be the splitting field of F where ζ is a root of F, and therefore a primitive p^rth root of unity.

(a) Suppose $r = 1$. Show that the discriminant of the basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{p-2}\}$ is equal to $\pm p^{p-2}$.

(b) Determine the sign in (a).

(c) Now for any r, show that the discriminant of the basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{\phi(p^r)-1}\}$ is equal to $\pm p^{p^{r-1}(pr-r-1)}$. (You will find it convenient to use the result of (a).)