# ALGEBRAIC NUMBER THEORY W4043

## 1. Homework, week 3, due September 27

1. Complete the proof of Lemma 4.9 of Hindry's book: we have proved the case where $K = \mathbb{Q}(\alpha)$, so you have to prove the case where $[K : \mathbb{Q}(\alpha)] = m > 1$.

2. Let $\mathcal{O}$ denote the ring of integers in $K = \mathbb{Q}(\sqrt{-14})$.

(a) Show that $3 + \sqrt{-14}$ is an irreducible element in $\mathcal{O}$.

(b) Show that 3 is not equal to $N_{K/\mathbb{Q}}(x)$ for any $x \in \mathcal{O}$.

(c) Show that 3 is an irreducible element in $\mathcal{O}$.

(d) Show that the principal ideal (3) is not a prime ideal and compute its factorization as a product of prime ideals.

3. Hindry's book, Exercise 6.16, p. 120. You can use Corollary 3-5.10 from Hindry's book; it will be proved later in the semester.

4. (a) Let $f(X) \in \mathbb{Q}[X]$ be a cubic polynomial, and let $K/\mathbb{Q}$ denote its splitting field. Suppose $[K : \mathbb{Q}] = 3$. Prove that all the roots of $f$ are real.

(b) Find a cubic polynomial $f(X) \in \mathbb{Q}[X]$ such that the splitting field $K/\mathbb{Q}$ is of degree 3, and such that the prime 5 is inert in $K$. What is the order of the residue field of $K$ at the unique prime dividing 5?