Homework, week 2, due September 20

Part I: Review of modules and Noetherian rings
This is background for Part II; exercises are not to be handed in!

1. Read and do all (or most of) the exercises on modules over a PID at http://www.imsc.res.in/~knr/14mayafs/Notes/ps.pdf

2. Study the notes on Noetherian rings and do all (or most of) the exercises at http://www.math.columbia.edu/~harris/W40432017/Harvardnotes.pdf
   (These notes were copied from an anonymous Harvard Mathematics Department website two years ago, but are no longer accessible).

4. Let $A$ be a Noetherian ring and let $f : A \to A$ be a ring homomorphism. Prove that $f$ is an isomorphism if and only if $f$ is surjective.
   (Hint. Assume that $f$ is surjective and denote by $I_j$ the kernel of $f^{(j)} = f \circ f \circ \cdots \circ f$ ($j$ times). Show that $\{I_j\}$ forms an increasing sequence of ideals of $A$ and therefore $I_j = I_{j+1}$ for some $j$. Deduce that $f(f^{(j)}(a)) = 0 \Rightarrow f^{(j)}(a) = 0$ for any $a \in A$, and use the surjectivity of $f$ to complete the proof.)
Part II: Exercises on Dedekind domains

1. Let $\mathcal{O}$ be the ring of integers of a number field $K$. A fractional ideal of $\mathcal{O}$ is an $\mathcal{O}$-submodule of $K$ of finite type. Let $M \subset K$ be a fractional ideal of $\mathcal{O}$.

(a) Show that there exists $r \in \mathcal{O}$ such that $rm \in \mathcal{O}$ for all $m \in M$.

(b) Show that if $M$ and $M'$ are fractional ideals then $M \cdot M'$, defined to be the $\mathcal{O}$-submodule of $K$ generated by products $m \cdot m'$, with $m \in M$ and $m' \in M'$, is again a fractional ideal.

(c) Show that if $M$ is a fractional ideal then $M^{-1}$, defined to be the $\mathcal{O}$-submodule of $a \in K$ such that $a \cdot m \in \mathcal{O}$ for all $m \in M$, is again a fractional ideal.

2. Prove the following Proposition:

Proposition. Let $\mathcal{O}$ be the ring of integers of a number field, $\{p_i, i \in \mathbb{N}\}$ a sequence of two-by-two distinct prime ideals. Then $\bigcap_i p_i = \{0\}$.

3. Let $R$ be an integral domain with fraction field $K$. A multiplicative subset $S \subset R$ is a subset such that,

- $1 \in S$, $0 \notin S$;
- If $s, s' \in S$ then $ss' \in S$.

The localization $S^{-1}R$ is the subset of $K$ consisting of elements $\frac{r}{s}$ with $r \in R$ and $s \in S$. (Alternatively, it is the set of equivalence classes of pairs $(r, s)$, with $r \in R$ and $s \in S$, with $(r, s)$ equivalent to $(r', s')$ if and only if $rs' = r's$).

(Localization is also defined for general commutative rings, but the definition is more elaborate.) After convincing yourself that $S^{-1}R$ is a ring, show that

(a) If $S$ is the set of non-zero elements of $R$, then $S^{-1}R = K$;
(b) If $R$ is a Dedekind domain, then so is $S^{-1}R$ for any multiplicative subset $S \subset R$.
(c) If $I \subset R$ is an ideal, let $S^{-1}I \subset S^{-1}R$ be the ideal of $S^{-1}R$ generated by $I$. Show that the map $I \mapsto S^{-1}I$ is a surjection from the set of ideals of $R$ to the set of ideals of $S^{-1}R$. Use the proof to construct a bijection between the set of prime ideals of $S^{-1}R$ and the subset of prime ideals $p \subset R$ such that $p \cap S = \emptyset$.
(d) Let $R$ be a Dedekind domain, $p \subset R$ be a prime ideal, let $S_p = R \setminus p$, and define $R_p = S_p^{-1}R$. Show that $R_p$ is a discrete valuation ring, i.e. a Dedekind domain with a unique non-zero prime ideal. In particular, show (using problem 2) that every non-zero element $a \in R_p$ has a unique factorization of the form $a = uc^b$, where $c$ is a generator of the unique non-zero prime ideal of $R_p$, $b$ is a non-negative integer, and $u$ is an invertible element of $R_p$. 
