ALGEBRAIC NUMBER THEORY W4043

Homework, week 2, due September 20

Part I: Review of modules and Noetherian rings

This is background for Part II; exercises are not to be handed in!

1. Read and do all (or most of) the exercises on modules over a PID at
`http://www.imsc.res.in/~knr/14mayafs/Notes/ps.pdf`

2. Study the notes on Noetherian rings and do all (or most of) the exercises at
`http://www.math.columbia.edu/~harris/W40432017/Harvardnotes.pdf`
(These notes were copied from an anonymous Harvard Mathematics Department website two years ago, but are no longer accessible).

4. Let A be a Noetherian ring and let $f : A \to A$ be a ring homomorphism. Prove that f is an isomorphism if and only if f is surjective.

(Hint. Assume that f is surjective and denote by I_j the kernel of $f^{(j)} = f \circ f \circ \cdots \circ f$ (j times). Show that $\{I_j\}$ forms an increasing sequence of ideals of A and therefore $I_j = I_{j+1}$ for some j Deduce that $f(f^{(j)}(a)) = 0 \Rightarrow f^{(j)}(a) = 0$ for any $a \in A$, and use the surjectivity of f to complete the proof.)

Part II: Exercises on Dedekind domains

1. Let $\mathcal{O}$ be the ring of integers of a number field K. A *fractional ideal* of $\mathcal{O}$ is an $\mathcal{O}$-submodule of K of finite type. Let $M \subset K$ be a fractional ideal of $\mathcal{O}$.

(a) Show that there exists $r \in \mathcal{O}$ such that $rm \in \mathcal{O}$ for all $m \in M$.

(b) Show that if M and M' are fractional ideals then $M \cdot M'$, defined to be the $\mathcal{O}$-submodule of K generated by products $m \cdot m'$, with $m \in M$ and $m' \in M'$, is again a fractional ideal.

(c) Show that if M is a fractional ideal then M^{-1}, defined to be the $\mathcal{O}$-submodule of $a \in K$ such that $a \cdot m \in \mathcal{O}$ for all $m \in M$, is again a fractional ideal.

2. Prove the following Proposition:

Proposition. Let $\mathcal{O}$ be the ring of integers of a number field, $\{\mathfrak{p}_i, i \in \mathbb{N}\}$ a sequence of two-by-two distinct prime ideals. Then $\cap_i \mathfrak{p}_i = \{0\}$.

3. Let R be an integral domain with fraction field K. A *multiplicative subset* $S \subset R$ is a subset such that,

- $1 \in S$, $0 \notin S$;
- If $s, s' \in S$ then $ss' \in S$.

The *localization* $S^{-1}R$ is the subset of K consisting of elements $\frac{r}{s}$ with $r \in R$ and $s \in S$. (Alternatively, it is the set of equivalence classes of pairs (r, s), with $r \in R$ and $s \in S$, with (r, s) equivalent to (r', s') if and only if $rs' = r's$).

(Localization is also defined for general commutative rings, but the definition is more elaborate.) After convincing yourself that $S^{-1}R$ is a ring, show that

(a) If S is the set of non-zero elements of R, then $S^{-1}R = K$;

(b) If R is a Dedekind domain, then so is $S^{-1}R$ for any multiplicative subset $S \subset R$.

(c) If $I \subset R$ is an ideal, let $S^{-1}I \subset S^{-1}R$ be the ideal of $S^{-1}R$ generated by I. Show that the map

$$I \mapsto S^{-1}I$$

is a surjection from the set of ideals of R to the set of ideals of $S^{-1}R$. Use the proof to construct a bijection between the set of prime ideals of $S^{-1}R$ and the subset of prime ideals $\mathfrak{p} \subset R$ such that $\mathfrak{p} \cap S = \emptyset$.

(d) Let R be a Dedekind domain, $\mathfrak{p} \subset R$ be a prime ideal, let $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$, and define $R_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}R$. Show that $R_{\mathfrak{p}}$ is a *discrete valuation ring*, i.e. a Dedekind domain with a unique non-zero prime ideal. In particular, show (using problem 2) that every non-zero element $a \in R_{\mathfrak{p}}$ has a unique factorization of the form $a = uc^b$, where c is a generator of the unique non-zero prime ideal of $R_{\mathfrak{p}}$, b is a non-negative integer, and u is an invertible element of $R_{\mathfrak{p}}$.