ALGEBRAIC NUMBER THEORY W4043

Homework, week 10, due November 29


2. An arithmetic function is a function \( f : \mathbb{N} \to \mathbb{C} \). An arithmetic function \( f \) is multiplicative if \( f(ab) = f(a)f(b) \) whenever \( a \) and \( b \) are relatively prime. Suppose \( f \) and \( g \) are two arithmetic functions. Define the convolution

\[
(f * g)(n) = \sum_{d|n} f(d)g(n/d).
\]

(a) Let \( \tau(n) \) denote the number of integers dividing \( n \). Let \( 1 \) be the function defined by \( 1(n) = 1 \) for all \( n \). Show that \( 1 * 1 = \tau \).

(b) Suppose \( f \) and \( g \) are multiplicative functions. Show that \( f * g \) is also multiplicative.

(c) Define the Möbius function \( \mu \) to be the unique multiplicative function such that \( \mu(1) = 1 \), \( \mu(p) = -1 \) for any prime \( p \), and \( \mu(n) = 0 \) if \( n \) is not square-free. Let \( f \) be the function \( f(n) = n \) for all \( n \). Compute \( f * \mu \).

(d) Define the von Mangoldt function \( \Lambda \) by \( \Lambda(1) = 1 \), \( \Lambda(p) = \log(p) \) if \( n = p^i \) for some prime \( p \), \( \Lambda(n) = 0 \) if \( n \) is not a prime power. Let

\[
D(s) = \sum_n \frac{\Lambda(n)}{n^s}.
\]

Show that \( D(s) \) converges absolutely for \( \text{Re}(s) > 1 \) and that, on the half plane \( \text{Re}(s) > 1 \), we have the equality

\[
D(s) = -\frac{\zeta'(s)}{\zeta(s)}
\]

where \( \zeta(s) = \sum_n \frac{1}{n^s} \) is the Riemann zeta function.