

ALGEBRAIC NUMBER THEORY W4043

HOMEWORK, WEEK 2, DUE SEPTEMBER 24

Part I: Review of modules and Noetherian rings

This is background for Part I; exercises are not to be handed in!

1. Read and do all (or most of) the exercises on modules over a PID at <http://www.imsc.res.in/~knr/14mayafs/Notes/ps.pdf>
2. Study the notes on Noetherian rings and do all (or most of) the exercises at <http://isites.harvard.edu/fs/docs/icb.topic1032186.files/week2.pdf>
3. Study exercises 1 and 3 on modules over Noetherian rings at <https://www.maths.nottingham.ac.uk/personal/ibf/als3/ex3.pdf>

Part II: Exercises on Dedekind domains

1. Let \mathcal{O} be the ring of integers of a number field K . A *fractional ideal* of \mathcal{O} is an \mathcal{O} -submodule of K of finite type. Let $M \subset K$ be a fractional ideal of \mathcal{O} . Show that there exists $r \in \mathcal{O}$ such that $rm \in \mathcal{O}$ for all $m \in M$.

2. Prove the following Proposition:

Proposition 0.1. *Let \mathcal{O} be the ring of integers of a number field, $\{\mathfrak{p}_i, i \in \mathbb{N}\}$ a sequence of two-by-two distinct prime ideals. Then $\bigcap_i \mathfrak{p}_i = \{0\}$.*

3. Let R be an integral domain with fraction field K . A *multiplicative subset* $S \subset R$ is a subset such that,

- $1 \in S, 0 \notin S$;
- If $s, s' \in S$ then $ss' \in S$.

The *localization* $S^{-1}R$ is the subset of K consisting of elements $\frac{r}{s}$ with $r \in R$ and $s \in S$. (Alternatively, it is the set of equivalence classes of pairs (r, s) , with $r \in R$ and $s \in S$, with (r, s) equivalent to (r', s') if and only if $rs' = r's$).

(Localization is also defined for general commutative rings, but the definition is more elaborate.) After convincing yourself that $S^{-1}R$ is a ring, show that

- (a) If S is the set of non-zero elements of R , then $S^{-1}R = K$;
- (b) If R is a Dedekind domain, then so is $S^{-1}R$ for any multiplicative subset $S \subset R$.
- (c) If $I \subset R$ is an ideal, let $S^{-1}I \subset S^{-1}R$ be the ideal of $S^{-1}R$ generated by I . Show that the map

$$I \mapsto S^{-1}I$$

is a surjection from the set of ideals of R to the set of ideals of $S^{-1}R$. Use the proof to construct a bijection between the set of prime ideals of $S^{-1}R$ and the subset of prime ideals $\mathfrak{p} \subset R$ such that $\mathfrak{p} \cap S = \emptyset$.

(d) Let R be a Dedekind domain, $\mathfrak{p} \subset R$ be a prime ideal, let $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$, and define $R_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}R$. Show that $R_{\mathfrak{p}}$ is a *discrete valuation ring*, i.e. a Dedekind domain with a unique non-zero prime ideal. In particular, show (using problem 2) that every non-zero element $a \in R_{\mathfrak{p}}$ has a unique factorization of the form $a = uc^b$, where c is a generator of the unique non-zero prime ideal of $R_{\mathfrak{p}}$, b is a non-negative integer, and u is an invertible element of $R_{\mathfrak{p}}$.