ALGEBRAIC NUMBER THEORY W4043

Take home final, due December 21, 2015

1. (a) Show that if $p \neq q$, then $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic as fields.

(b) Show that any field automorphism of $\mathbb{Q}_p$ is continuous. Then show that the only automorphism of $\mathbb{Q}_p$ is the identity.

2. We will construct number fields whose integer rings are not generated by 1, 2, or 3 elements.

(a) Let K be a number field, $[K:\mathbb{Q}] = n$, $\mathcal{O}_K \subset K$ its ring of integers. Suppose 2 splits completely in $\mathcal{O}_K$:

$$(2)\mathcal{O}_K = \prod_{i=1}^{n} \mathfrak{p}_i$$

where each $\mathfrak{p}_i$ is a prime ideal. Show that $\mathcal{O}_K/(2)\mathcal{O}_K = \prod_{i=1}^{n} \mathbb{F}_2$.

(b) Let d be a positive integer. Show that there are exactly 2^d distinct ring homomorphisms

$$\mathbb{F}_2[X_1, X_2, \ldots, X_d] \to \mathbb{F}_2.$$

Deduce that, in the notation of (a), if $\mathcal{O}_K$ has d generators over $\mathbb{Z}$, then $[K:\mathbb{Q}] \leq 2^d$.

(3) Let ζ be a 151-th root of 1, $L = \mathbb{Q}(\zeta)$. Show that the cyclotomic field L contains a unique subfield K of degree 10 over $\mathbb{Q}$. (Check, but don't bother writing down, that 151 is a prime number.) Show that

$$2^{15} \equiv 1 \pmod{151}.$$

Conclude that $\mathcal{O}_K$ has more than 3 generators over $\mathbb{Z}$.

3. (a) Which prime numbers can be written in the form $m^2 + mn + 2n^2$, with m and n non-negative integers? In how many ways? (You will need to use Minkowski's bound to show that a certain field has class number 1.) Which integers can be written in that form?

(b) Consider the series

$$D(s) = \sum_{m \geq 0, n \geq 0, (m,n) \neq (0,0)} \frac{1}{(m^2 + mn + 2n^2)^s}$$

as a function of $s \in \mathbb{C}$. Determine the set of s for which $D(s)$ converges absolutely. Does $D(s)$ have an Euler product?

4. Let $d = \prod_{i=1}^{r} p_i$ be a product of distinct odd prime numbers, with $d \equiv 1 \pmod 4$. Let $K = \mathbb{Q}(\sqrt{d})$, and for each i, let $\mathfrak{p}_i$ be the unique prime

ideal in $\mathcal{O}_K$ containing p_i. Show that there is an imaginary quadratic field $L/\mathbb{Q}$ such that, for every i, the ideal $\mathfrak{p}_i\mathcal{O}_L \subset \mathcal{O}_L$ is a product of two distinct prime ideals.