V2000: Notes for Week 2

Reading assignment

Dumas-McCarthy: (DM) Ch 1, sections 1.3 – 1.7, 2.1 – 2.3
Daepp-Gorkin (DG): Ch 10, 14, 4 (All readings from Daepp-Gorkin are optional.)

1. Functions

Last week we discussed what we might call the *static* properties of sets: identity, inclusion, unions, intersections, and so on. This week we turn to their *dynamic* properties: how they relate to one another. Let X and Y be sets.

Definition 1.1. *A function f from X to Y (writtn $f : X \to Y$) is an assignment to every element $x \in X$ of an element $f(x) \in Y$. X is called the* **domain** *of f and Y is called the* **codomain** *or* **range** *of f. (In (DG), the word "range" is used to denote a subset of Y, see below.)*

(In (DM) the notation $f(x)$ is used in Example 1.9 though it has not been defined.)

The word "assignment" in the definition is fraught. Calculus (or more precisely *real analysis*) is the theory of functions from $\mathbb{R}$ to $\mathbb{R}$. In this setting the notion of function, and the notation $y = f(x)$, is exceedingly familiar, because one "assigns" $f(x)$ to x by writing a formula for $f(x)$. For example, $f(x) = e^{cos(x)}$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Similarly $f(x) = tan(x)$ is a function from $(-\pi/2, \pi/2)$ to $\mathbb{R}$ – but not from $\mathbb{R}$ to $\mathbb{R}$, because it is undefined at values of x where $cos(x) = 0$.

One can do arithmetic with functions from $\mathbb{R}$ to $\mathbb{R}$: if f and g are functions from $\mathbb{R}$ to $\mathbb{R}$, then so are $f + g$, $f - g$, and $f \cdot g$. If $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$, then $f/g : \mathbb{R} \to \mathbb{R}$ is a function; otherwise, it is a function from X to $\mathbb{R}$, where $X = \{x \in \mathbb{R} \mid g(x) \neq 0\}$. It makes no sense to do arithmetic with functions between general sets, but some of the notions associated with $\mathbb{R}$-valued functions generalizes.

Definition 1.2. *Let $f : X \to Y$ be a function. The* **graph** *of f, denoted Γ_f, is a subset of the Cartesian product $X \times Y$ given by $\{(x, y) \mid x \in X, y = f(x)\}$.*

There is a statement hidden in this definition: Let $(x, y) \in X \times Y$. Then $(x, y) \in \Gamma_f$ if and only if $y = f(x)$.

Proposition 1.3. *Let $Z \subset X \times Y$ be a subset. Then Z is the graph of a function if and only if*

(i) *If $x \in X$, there is some $y \in Y$ such that $(x, y) \in Z$;*
(ii) *If $(x, y) \in Z$ and $(x, y') \in Z$ then $y = y'$.*

Proof. Suppose Z is the graph of a function. Then the two properties are obvious: we have $y = f(x)$ and $y' = f(x)$ so $y = y'$. Conversely, if Z satisfies (i) and (ii), let $f(x)$ be the y such that $(x, y) \in Z$. Such a y exists by (i) and is unique by (ii). So f is a function. $\square$

The properties can be expressed in terms of some useful terminology.

Definition 1.4. *Let $f : X \to Y$ be a function. Then*

(a) $f : X \to Y$ is **surjective** *if for all $y \in Y$ there is $x \in X$ such that $f(x) = y$.*

(b) $f : X \to Y$ is **injective** *if $x_1 \neq x_2 \in X$ implies $f(x_1) \neq f(x_2)$, ie. distinct points in X have distinct images.*

(c) $f : X \to Y$ is **bijective** *if it is both injective and surjective.*

Another way of saying this: f is surjective if for all $y \in Y$ the equation $f(x) = y$ has **at least one** *solution. f is injective if for all y this equation has* **at most one** *solution, i.e. $f(x_1) = f(x_2)$ implies $x_1 = x_2$ (the contrapositive of the initial definition).*

To explain the relevance of these definitions, here are some more definitions. Define the *projection* $p_X : X \times Y \to X$ by $p_X((x, y)) = x$. This is a function!

Let $Z \subset Z'$ be a subset. Let $\iota_Z : Z \to Z'$ be the inclusion map: $\iota(z) = z$ where the first z is considered as an element of Z and the second as an element of z'. This is a function!

Let $f : X \to Y$, $g : Y \to Z$ be functions. Let $g \circ f : X \to Z$ be the composite function: $g \circ f(x) = g(f(x))$.

Proposition 1.5. *Let $Z \subset X \times Y$ be a subset. Then Z is the graph of a function if and only if the composite $p_X \circ \iota_Z : Z \to X$ is bijective, where $p_X : X \times Y \to X$ is the projection and $\iota_Z : Z \to X \times Y$ is the inclusion.*

Remark 1.6. *We will see that for any pair of sets X and Y, the collection of all functions from X to Y is a naturally a set. Indeed, the collection of all subsets of $X \times Y$ is a set, and the collection of graphs of functions is a subset of this set.*

Examples of injective, surjective, bijective functions. (i) $\cos : \mathbb{R} \to \mathbb{R}$ is neither injective nor surjective.

(ii) $f(x) = x^2$ is neither injective not surjective as a function from $\mathbb{R}$ to $\mathbb{R}$. But as a function from $\mathbb{R}^+$ to $\mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$, it is bijective. The same holds for any even power; if $n \in \mathbb{N}$ is odd then $f(x) = x^n$ is bijective from $\mathbb{R}$ to $\mathbb{R}$.

(iii) $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection. This allows us to define the function arctan: for any $y \in \mathbb{R}$, $\arctan(y)$ is the element $x \in (-\pi/2, \pi/2)$ such that $\tan(x) = y$. This is *well-defined* because (a) there is such an x (because tan is surjective) and (b) there is exactly one such x (because tan is injective).

(iv) Let F be the set of functions from X to Y and let G be the set of $Z \subset X \times Y$ satisfying the two conditions of Proposition 1.3. Then the assignment $f \mapsto \Gamma_f$ is a bijection between F and G. (This is a reinterpretation of Proposition 1.3, but it only makes sense when you know why F and G are sets.)

1.1. **Images and inverse images.** Example (iii) above illustrates the definition of *inverse functions*. If $f : X \to Y$ is bijective, we can define $f^{-1} : Y \to X$ by saying

$f^{-1}(y)$ is the unique $x \in X$ such that $f(x) = y$. This is an elaborate definition, and it may be best to place it in the context of general images and inverse images.

Definition 1.7. *Let $f : X \to Y$ be a function. Let $U \subset X$ and $V \subset Y$ be subsets.*
(a) For $x \in X$, the **image** *of x under f, or $f(x)$, is the value $f(x) \in Y$.*
(b) The **image** *of U under f (denoted $f(U)$ – or $f[U]$ in (DG)) is the subset $\{f(x) \in Y \mid x \in U\}$, or equivalently (and more usefully) $f(U)$ is the set of $y \in Y$ such that there exists $x \in U$ for which $f(x) = y$.*
(c) For $y \in Y$, the **inverse image** *(or* **preimage***) of y under f, or $f^{-1}(y)$ is the set $\{x \in X \mid f(x) = y\}$. This is in general a subset with more than one element.*
(d) The **inverse image** *(or* **preimage***) of V under f (denoted $f^{-1}(V)$ – or $f^{-1}[V]$ in (DG)) is the subset $\{x \in X \mid f(x) \in V\}$.*

If f is bijective, then the same notation $f^{-1}(y)$ is used to denote the inverse image of the element $y \in Y$ (or of the subset $\{y\} \subset Y$) and the image of y under the inverse function f^{-1}. In general, f^{-1} is not the name of a function, because $f^{-1}(y)$ may consist of more than one element! For example, if $Y = p$ is a single point, there is a unique function $f : X \to Y$, namely $f(x) = p$ for all $x \in X$. Then $f^{-1}(p) = X$. (More generally, $f^{-1}(Y) = X$, always.)

Example 1.8. *Let $X = Y = \mathbb{Z}$, $f(x) = x^2 - 4$. Then $f^{-1}(0) = \{-2, 2\}$ has two elements. $f^{-1}(-5) = \emptyset$. $f^{-1}(-4) = \{0\}$. In general, the set $f^{-1}(y)$ has either 0, 1, or 2 elements.*

The next Propositions should be obvious.

Proposition 1.9. *Let $f : X \to Y$ be a function.*
(a) f is surjective if and only if $f(X) = Y$.
(b) f is injective if and only if, for all $y \in Y$, $f^{-1}(y)$ has at most one element.

For any set Z, the *identity function* $Id_Z : Z \to Z$ is the function such that $Id_Z(z) = z$ for all z.

Proposition 1.10. *Let $f : X \to Y$ be a bijection. Then $f^{-1} \circ f = Id_X$, $f \circ f^{-1} = Id_Y$.*

1.2. **Sequences.** Notation for infinite sequences, unions, intersections as in (DG).
We can write $n \mapsto x_n$ in function notation: $f : \mathbb{N} \to X$ with $f(n) = x_n$.
Sequences of real numbers ($X = \mathbb{R}$) will be covered starting in chapter 5. Familiar examples: $f(n) = t^n$ for some $t \in \mathbb{R}$; $f(n) = \frac{1}{(n+1)^s}$ for s a real number, $f(n) = \sum_{i=1}^{n} \frac{1}{2^i}$, etc. These sequences are given by formulas (rules). One of the themes of the last part of the course is that most sequences cannot be given by rules, where "most" and "rules" have precise meanings.

2. Relations

2.1. **Definition of relations.**

Definition 2.1. *Let X and Y be sets. A* **relation** *from X to Y is a subset $R \subset X \times Y$. If $X = Y$, we say R is a* **relation on** *X.*

In other words, a relation from X to Y is a collection of ordered pairs (x, y) where $x \in X$ and $y \in Y$. Obviously a relation from X to Y can also be seen as a relation from Y to X, by reversing the order.

2.1.1. *Relations between different sets.* If $f : X \to Y$ is a function, then $\Gamma_f \subset X \times Y$ is a relation, and any relation with properties (i) and (ii) of 1.3 is the graph of a function.

There are no other examples in the book. Although the case $X = Y$ is the most important, here are a few examples.

Rivers. X is the set of continents, Y is the set of rivers, and xRy is the relation "y is contained in x.

Shoes. X is the set of boxes, Y is the set of shoes, and xRy is again the relation "y is contained in x.

Birthdays. X is the set of people, Y is the set of dates in the 21st century, and xRy is the relation "y is x's birthday.

In the example of Shoes, R is not the graph of a function from Y to X (not every shoe is in a box). In the example of Birthdays, R is the graph of a function f from X to Y (everyone has exactly one birthday); moreover, the function f is obviously surjective (someone is born every day) but not injective.

2.1.2. *Relations on a set X.* There are examples in the book of relations on X. We will consider order relations (like $<$, $\leq$, $>$, $\geq$) in the next section. Here are some more exotic examples of relations on X.

Intersection. Let X be the set of closed intervals in $\mathbb{R}$. For $x, y \in X$, say xRy if $x \cap y \neq \emptyset$.

Non-intersection. Let X be the set of closed intervals in $\mathbb{R}$. For $x, y \in X$, say xRy if $x \cap y = \emptyset$.

Union. Let X be the set of closed intervals in $\mathbb{R}$. For $x, y \in X$, say xRy if $[0, 1] \subset x \cup y$. (Thus if $x = [0, \frac{1}{2}]$ and $y = [\frac{1}{3}, 2]$, then xRy, but not if $y = [\frac{1}{3}, \frac{2}{3}]$.)

2008. Let X be the set of states of the United States. For $x, y \in X$, say xRy if x and y gave their electoral votes to different candidates in the 2008 presidential election.

Socks. Let X be the set of all socks. For $x, y \in X$, say xRy if x and y have the same color.

The following properties of relations are particularly important.

Definition 2.2. *Let X be a set, R a relation on X.*
(a) R is **reflexive** *if for all $x \in X$, xRx.*
(b) R is **symmetric** *if for all $x, y \in X$, xRy implies yRx.*
(c) R is **antisymmetric** *if for all $x, y \in X$, xRy and yRx implies $x = y$.*
(d) R is **transitive** *if for all $x, y, z \in X$, xRy and yRz implies xRz.*
A relation R is an **equivalence relation** *if it satisfies (a), (b), and (d).*

Which of these properties apply to the examples above?

2.2. **Orderings.**

Definition 2.3. *Let X be a set, R a relation on X. We say R is a* **partial ordering** *if it is reflexive, antisymmetric, and transitive.*

We say that R is a **linear ordering** *if in addition, for any $x, y \in X$, either xRy or yRx (not both unless $x = y$.*

Important examples: inclusion of sets, chronological order of historical events, $<$, $\leq$, $>$, $\geq$, divisibility of integers (in the book). Divisibility is *not* a linear ordering, nor is $<$.

A more familiar example: alphabetical (lexicographic) ordering. One can imagine two ways to order words in a dictionary. The easiest way to define would be this: the word $\mathfrak{A}$ comes before the word $\mathfrak{B}$ if the first letter of $\mathfrak{A}$ comes before or equals the first letter of $\mathfrak{B}$, the second letter of $\mathfrak{A}$ comes before or equals the second letter of $\mathfrak{B}$. This is definitely a partial ordering (check this) but it is not a linear ordering (does *bird* come before or after *reptile*), and a dictionary is not very useful unless the words can be placed in linear order.

The formal definition of lexicographic ordering is surprisingly intricate (see example 2.8 in the book). In fact, suppose X is any set and R is a linear ordering on the set X. Let Y be the set of finite sequences of elements of X. This can be defined as a set of sequences if we let $X* = X \cup *$, where $*$ is a blank, and we let Y be the set of functions $f : \mathbb{N} \rightarrow X*$ such that if $f(n) = *$ then $f(m) = *$ whenever $m \geq n$. Let $f, g \in Y$. Define the relation S on Y as follows. Let $k \in \mathbb{N}$ be the smallest integer such that $f(k) \neq g(k)$, and say fSg if $f(k)Rg(k)$.

Example 2.9 in the book (comparing apples and oranges) is another example of this kind.