V2000: Notes for Week 1

Reading assignment

Dumas-McCarthy: (DM) Ch 0, Ch 1, sections 1 and 2
Daepp-Gorkin (DG): Ch 1,2,6
The online description of the course (at `http://www.math.columbia.edu/~harris/MathV2000.htm`)

1. Statements (Propositions)

The first few days of the course are devoted to getting used to the *language* of mathematics. Mathematical language is composed of *sentences*, which are called *statements* (more formally *proposition*) to distinguish them from the sentences of natural language. In order to qualify as a statement (a mathematical proposition), a sentence must be either **true** or **false**. However, it is still a sentence constructed out of words and symbols that can be interpreted as words.

Question 1.1. *Which of these are statements? (adapted from DG)*

(a) *All the world's a stage.*
(b) *Two plus three equals six.*
(c) *The number of students in the class is a positive integer.*
(d) $X + 3 = 0$.
(e) *Every sentence on this page is false.*
(f) *April is the cruelest month.*

Much of analytic philosophy is concerned with distinguishing statements from sentences with no meaningful truth value. This is not a course in philosophy; our goal is to illustrate the use of sentences as mathematicians use them. So we will not be contending with the ambiguity of sentences (a) and (f); instead, we will develop a vocabulary of words and symbols whose terms are unambiguous. Integers and real numbers will be part of this vocabulary (though the precise meaning of integers and real numbers is also of great interest to philosophers). *True* and *False* will eventually be used in a purely technical sense, but for the time being they will be used informally.

1.1. Operations with statements, propositional connectives.

Given a statement, you can make new statements by combining them.

$\neg P$ means NOT P (this has the opposite truth values to P) – also written $\sim P$

$P \wedge Q$ means P and Q,

$P \vee Q$ means P or Q. (nonexclusive OR: $P \vee Q$ is true if either P or Q or both are true.)

$P \Rightarrow Q$ means "P implies Q".

The symbols $\neg, \wedge, \vee, \Rightarrow$ are the basic *propositional connectives*. These are studied in more detail in subsequent weeks (see (DM), Chapter 3).

Exercise 1.2. *Explain with a few examples why* $P \Rightarrow Q$ *has the same meaning as* $Q \vee \neg P$.

$P \Leftrightarrow Q$ means "P is equivalent to Q", i.e. P holds if and only if Q holds – the two statements have the same truth values.

These new statements are defined by *truth tables*. For example:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Thus $P \Rightarrow Q$ is false ONLY IF P holds and Q does not: if P is false $P \Rightarrow Q$ is true. Therefore, knowing that $P \Rightarrow Q$ is true, while P is false gives NO INFORMATION on Q: it could be true or false.

Exercise 1.3. *(optional) Redo Exercise 1.2 by showing that the two expressions* $P \Rightarrow Q$ *and* $Q \vee \neg P$ *have the same truth tables.*

A proof strings together a series of statements and logical reasoning using truth tables. The standard syllogism (*modus ponens*) is: P holds and $P \Rightarrow Q$ holds, therefore Q holds. So we advance from knowing that P is true to knowing that Q is true; we say we *derive* the truth of Q from that of P.

Example: When it rains, the roads are slippery. It is raining. Therefore the roads are slippery.

Here P is "It is raining"; Q is "The roads are slippery".

Exercise 1.4. *Explain what the following statements mean as simply as you can:*
a) $(\neg P) \vee (\neg Q)$
b) $\neg(P \Rightarrow Q)$.

Propositional calculus poems. Someone seems to think there ought to be poems written in the *Propositional Calculus*, which is the name for the branch of logic devoted to reasoning with formal propositions and connectives, as above. See `http://www.poetrysoup.com/poems/propositional_calculus`. However, no examples are given.

2. Sets

Much if not most of contemporary mathematics is expressed in the language of sets, which was introduced by Georg Cantor in the late 19th century. This can lead to some sticky philosophical issues, because a set, like a point or a line in Euclidean geometry, is a primitive notion; it doesn't have a meaningful definition. Euclid defines a point as "that which has no part" – which is intuitively compelling but not very precise. Similarly, one thinks of a set as a collection of objects, one talks about them as one does about any collections of objects, and one doesn't worry what kind of objects they are or where they come from. On the other hand, there are rules for forming new sets out of old sets – some of them strict rules, to avoid some of the more obvious philosophical paradoxes.

In practice one works with examples and in that way develops a feeling for what one can and can't do with sets. Fortunately, much of this work has taken place long before one encounters the formal notion of a set. Here are some examples from ordinary life, from elementary mathematics, and from less elementary mathematics.

Example 2.1. (a) *The candidates in the Republican presidential primaries.*
(b) *The molecules in the universe.*
(c) *The sides of a polygon.*
(d) *The solutions to a quadratic equation.*
(e) $\mathbb{Z}$ *(the integers).*
(f) $\mathbb{R}$ *(the real numbers).*
(g) *The solutions to a system of k linear equations in n variables.*
(h) *The empty set $\emptyset$.*

Given a set S and an object (or element a) **all** we can say is whether a is in S ($a \in S$) or a is not in S ($x \notin X$). The inclusion of objects in sets is closely related to the truth or falsity of a proposition. Recall the example

$$X + 3 = 0.$$

We said this was not a statement because its truth or falsity depends on X. We can define a set S to be the set of integers $X \in \mathbb{Z}$ such that the statement $X + 3 = 0$ is true. This is the usual way to get interesting sets in mathematics: starting from a sentence that depends on a variable in a set, look for the values of the variable that make the sentence true.

More formally, sets X can be defined by sentences like: $X = \{n \in \mathbb{Z} : P(n)\}$, where P is a statement depending on n. Here $\mathbb{Z}$ can be replaced by $\mathbb{R}$ or any other reference set.

To be is to be the value of a variable. (W. V. O. Quine, taken slighly out of context.)

Example 2.2. $X = \{n \in \mathbb{Z} : 3 \textit{ divides } n)\}$

The book has some idiosyncratic notation:
$\mathbb{N} = \big\{k \in \mathbb{Z} : k \geq 0\} = \{0, 1, 2, 3, \dots\big\}$.
$\lceil n \rceil = \{0, 1, \dots, n-1\}$. (thus the set $\lceil n \rceil$ contains exactly n elements.)

2.1. **Relations between sets.**

- **equality** The set X is entirely described by the set of elements that are in it. So the statement $X = Y$ means that an element a belongs to X if and only if it belongs to Y: in symbols
$$a \in X \iff a \in Y.$$
The technical name for this is **extensionality**; in philosophy the notion is traced back to Leibniz – long before Cantor – and it's called the *Identity of indiscernables*. Say X is the set of your properties, and Y is the set of your neighbor's properties. If all of your properties are also properties of your neighbor, and vice versa, then you and your neighbor are the same person – you are indiscernable.
- **subset** $A \subseteq B$ (same as $B \supseteq A$) – sometimes people write $A \subset B$ for this, but it is not so clear.
- **proper subset** $\emptyset \neq A \subsetneq B$ (same as $B \supsetneq A \neq \emptyset$)

Exercise 2.3. *Look at your neighbor, and use extensionality to determine whether or not you and your neighbor are the same person.*

2.2. **Operations on sets.** In other words, making new sets from old. (Details in class or in the book.)

- **union** $A \cup B$,
- **intersection** $A \cap B$, – also A, B are **disjoint** if $A \cap B = \emptyset$
- **complement** in U: if $X \subset U$ (here U is the "universal set") then $X^c = \{x \in U : x \notin X\}$.
- **Cartesian product** $A \times B$ (eg the plane $\mathbb{R}^2$).

DISTRIBUTIVE LAWS:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

DE MORGAN'S RULES: If $A, B \subset U$ then

- $(A \cup B)^c = A^c \cap B^c$,
- $(A \cap B)^c = A^c \cup B^c$

Exercise 2.4. *Prove the first of De Morgan's Rules. (To be discussed next time.)*

Here is a sample proof.

Proposition 2.5. *Let A, B, C be any sets. Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

Strategy: We must show that every element x in $A \cup (B \cap C)$ is also an element in $(A \cup B) \cap (A \cup C)$, and also that every element in $(A \cup B) \cap (A \cup C)$ is also an element in $A \cup (B \cap C)$. This is simple enough – we just do it.

Proof. i) We show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ so that $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$ then $x \in B$ and $x \in C$, so that $x \in A \cup B$ and $x \in A \cup C$. Hence $x \in (A \cup B) \cap (A \cup C)$. Therefore in both cases $x \in (A \cup B) \cap (A \cup C)$. This proves (i).

ii) We show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$, so we are done. If $x \notin A$ then we must have $x \in B$ and $x \in C$, i.e. $x \in B \cap C$. Therefore again $x \in A \cup (B \cap C)$.

This completes the proof. □

Typically, to prove results about sets you must make arguments about the properties of the *elements* of these sets.