PRACTICE FINAL SOLUTIONS

FALL 2018

UN1201: CALCULUS III

Problem 1.

(a) True, $\vec{u} \cdot (2\vec{v} - \vec{w}) = 2\vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w} = 2(1) - (2) = 0$.

(b) True, reparametrize the curve $\vec{r}(t) = (x, f(x))$ according to arc length so $0 = \kappa = |\vec{r}''(s)|$. Then $\vec{r}''(s) = 0$, so $r'(s)$ is constant. Then $\vec{r}(s) = \vec{r_0} + s\vec{u}$ for some vector $\vec{r_0}$ and unit vector $\vec{u}$.

(c) False: if S is a level surface of a function g, then the gradient of f is parallel to the gradient of g at the point P. Thus the gradient of f is perpendicular, not parallel, to the tangent plane at P.

(d) True, observe that $x^4 + y^4 - x^2y^2 = (x^4 - 2x^2y^2 + y^4) + x^2y^2 = (x^2 - y^2)^2 + (xy)^2 \geq 0$ for all real (x, y).

Problem 2.

(a) Let (x_0, y_0, z_0) be the intersection point of ℓ and m. From $\frac{z_0-2}{c} + 3 = x_0 = \frac{1-z_0}{3} + 2$, we deduce $3z_0 - 6 + 3c = c - cz_0$ and so $(3+c)z_0 + (-6+2c) = 0$. Then $z_0 = \frac{6-2c}{3+c}$. From $3\frac{z_0-2}{c} - 4 = y_0 = -2\frac{1-z_0}{3} + 3$, we deduce $9z_0 - 18 = -2c + 2cz_0 + 21c$ and so $(9-2c)z_0 + (-18-19c) = 0$. Then $\frac{6-2c}{3+c} = z_0 = \frac{18+19c}{9-2c}$, so

$$0 = (6-2c)(9-2c) - (18+19c)(3+c) = 4c^2 - 30c + 54 - 19c^2 - 75c - 54 = -15c^2 - 105c = -15c(c+7).$$

Being the denominator of $\frac{z-2}{c}$, we know that c must be nonzero. Then we must have $c = -7$.

(b) Let $t = x - 2 = \frac{3-y}{2} = \frac{1-z}{3}$, so ℓ is given by $x = t + 2, y = -2t + 3, z = -3t + 1$, i.e. $(1, -2, -3)t + (2, 3, 1)$. Similarly, letting $s = x - 3 = \frac{y+4}{3} = \frac{z-2}{-7}$, we have that m is given by $(1, 3, -7)t + (3, -4, 2)$.

Then the plane containing both lines is perpendicular to $(1, -2, -3) \times (1, 3, -7) = (23, 4, 5)$. From our expressions in (a), we know that $z_0 = \frac{6-2(-7)}{3+(-7)} = \frac{20}{-4} = -5$. Then $x_0 = \frac{1-z_0}{3} + 2 = \frac{1-(-5)}{3} + 2 = 4$ and $y_0 = -2(x_0 - 2) + 3 = -2(4-2) + 3 = -1$.

Date: December 6, 2018.

Hence $(4, -1, -5)$ is the point of intersection of ℓ and m (check that this is true by verifying that both symmetric equations hold for this point!).

The plane containing both lines can be written as

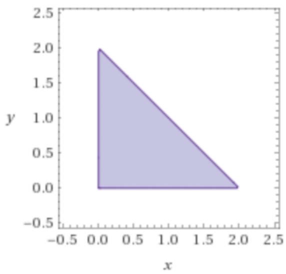
$$23(x - 4) + 4(y + 1) + 5(z + 5) = 0,$$

or as

$$23x + 4y + 5z - 63 = 0.$$

Problem 3.

(a) A sketch should look like:



(b) The first partials are $f_x(x, y) = 3x^2 - y - 1$ and $f_y(x, y) = -x + 2y$. If $0 = f_y(x, y) = -x + 2y$, then $x = 2y$, so $0 = f_x(x, y) = 3(2y)^2 - y - 1 = 12y^2 - y - 1$ which has solutions $-\frac{1}{4}$ and $\frac{1}{3}$. Since $x = 2y$, the two critical points of f are $(-\frac{1}{2}, -\frac{1}{4})$ and $(\frac{2}{3}, \frac{1}{3})$. We see that $(\frac{2}{3}, \frac{1}{3})$ lies in the region R since $\frac{1}{3} \geq 0$, $\frac{2}{3} \geq 0$, and $\frac{1}{3} + \frac{2}{3} = 1 \leq 2$. This is the only critical point of f in R, since $-\frac{1}{2} < 0$.

(c) The second partials are $f_{xx}(x, y) = 6x, f_{xy}(x, y) = -1, f_{yy}(x, y) = 2$. At the critical point $(\frac{2}{3}, \frac{1}{3})$, we have that $D = (6\frac{2}{3})(2) - (-1)^2 = 8 - 1 = 7 > 0$ with $f_{xx}(\frac{2}{3}, \frac{1}{3}) = 4 > 0$ so it is a local minimum for f.

We must also check f along the boundary of R, i.e. where $x = 0$, $y = 0$, or $x + y = 2$. When $x = 0$, we have $f(0, y) = y^2$ which clearly has a minimum of 0 and a maximum of 4 when restricted to $x = 0$ and R. When $y = 0$, we have $f(x, 0) = x^3 - x$ which has a minimum of $-\frac{2\sqrt{3}}{9}$ (differentiate with respect to x to find the single-variable minimum at $x = \frac{\sqrt{3}}{3}$) and a maximum of 6 (at $x = 2$) along $y = 0$ in R. When $x + y = 2$, we have $y = 2 - x$. Then $f(x, 2 - x) = x^3 - x(2 - x) + (2 - x)^2 - x = x^3 + 2x^2 - 7x + 4$. Using single-variable techniques, we have that f has a minimum of 0 (at $x = 1$) and a maximum of 6 (at $x = 2$) along $y = 2 - x$ in R.

The only critical point of f in R is $(\frac{2}{3}, \frac{1}{3})$ with local minimum $f(\frac{2}{3}, \frac{1}{3}) = -\frac{13}{27}$, which is less than the minimum value of $-\frac{2\sqrt{3}}{9}$ along the boundary of R, so f attains its absolute minimum in R of $-\frac{13}{27}$ at $(\frac{2}{3}, \frac{1}{3})$. Since there is no local maximum of f in the interior of R, the absolute maximum of f is its maximum along the boundary, which is the value of 6 attained at $(2, 0)$.

Problem 4.

(a) The level surfaces are of the form $k - 14 = x^2 - 2y^2 + \frac{z^2}{9}$. This is a hyperboloid of one sheet when $k - 14 > 0$, i.e. when $k > 14$, and a hyperboloid of two sheets when $k - 14 < 0$, i.e. when $k < 14$. When $k = 14$, we have an elliptic cone.

(b) The given level surface is $F(x, y, z) := x^2 - 2y^2 + \frac{z^2}{9} - 2 = 0$. Then $F_x(x, y, z) = 2x, F_y(x, y, z) = -4y, F_z(x, y, z) = \frac{2z}{9}$. Then the tangent plane to the given level surface at $(1, 2, 9)$ is given by $F_x(1, 2, 9)(x-1)+F_y(1, 2, 9)(y-2)+F_z(1, 2, 9)(z-9) = 0$, so it is

$$2(x - 1) + -8(y - 2) + 2(z - 9) = 0,$$

which can also be written as

$$2x - 8y + 2z - 4 = 0.$$

Problem 5.

(a) Along $\{y = 0\}$, we have $\lim_{(x,z)\to(0,0)} f(x, 0, z) = \lim_{(x,z)\to(0,0)} \frac{x^2+z^2}{x^2+z^2} = 1$. However, along $\{x = 0, z = 0\}$, we have $\lim_{y\to 0} f(0, y, 0) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$. Hence, the limit $\lim_{(x,y,z)\to(0,0,0)} f(x, y, z)$ does not exist.

Switching to spherical coordinates, we have that

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,0)} g(x, y, z) &= \lim_{\rho\to 0} g(\rho, \theta, \phi) \\
&= \lim_{\rho\to 0} \frac{\rho^4 \cos^4\theta \sin^4\phi + \rho^4 \sin^4\theta \sin^4\phi + \rho^4 \cos^4\phi}{\rho^2} \\
&= \lim_{\rho\to 0} \rho^2(\cos^4\theta \sin^4\phi + \sin^4\theta \sin^4\phi + \cos^4\phi) \\
&= 0.
\end{aligned}$$

(b) In order to be sure that the bridge will not collapse, we must have that the absolute maximum of w in the region given by $14.86 \le x \le 15.14$, $3.86 \le y \le 4.14$, and $2.86 \le z \le 3.14$ is less than 22.5. However, we can observe that w always increases with respect to x, y, and z. Thus, we only need to ensure that $w(15.14, 4.14, 3.14) > 22.5$.

Without directly calculating $w(15.14, 4.14, 3.14)$, we can try a linear approximation using the tangent plane at $(15, 4, 3)$ to overestimate $w(15.14, 4.14, 3.14)$. Calculate that $w(15, 4, 3) = 21, w_x(15, 4, 3) = \frac{33}{35}, w_y(15, 4, 3) = \frac{99}{28}$, and $w_z(15, 4, 3) = \frac{15}{7}$. Then the linear approximation is

$$\begin{aligned}
w(15.14, 4.14, 3.14) &\approx w(15, 4, 3) + w_x(15, 4, 3)(0.14) + w_y(15, 4, 3)(0.14) + w_z(15, 4, 3)(0.14) \\
&= 21 + (\frac{33}{35} + \frac{99}{28} + \frac{15}{7})(0.14) \\
&\leq 21 + (1 + 4 + 3)(0.14) \\
&= 21 + 8(\frac{7}{50}) \\
&= 21 + \frac{56}{50} \\
&= 22.12 \\
&< 22.5.
\end{aligned}$$

However, we do not know if the linear approximation overestimates or underestimates the value of $w(15, 4, 3)$, so we cannot be absolutely sure.

Problem 6.

(a) Observe that $\vec{r}(1) = (2, 1, 0) = P$ and $\vec{r}(2) = (4, 4, \ln(2)) = Q$. Using the components of $\vec{r}'(t) = (2, 2t, \frac{1}{t})$, the arc length of C between P and Q is

$$\begin{aligned}
L &= \int_1^2 \sqrt{4 + 4t^2 + \frac{1}{t^2}} dt \\
&= \int_1^2 \sqrt{(2t + \frac{1}{t})^2} dt \\
&= \int_1^2 (2t + \frac{1}{t}) dt \\
&= t^2 + \ln(t) \,|_1^2 \\
&= (4 - 1) + (\ln(2) - 0) \\
&= 3 + \ln(2).
\end{aligned}$$

(b) Compute that $\vec{r}''(t) = (0, 2, -\frac{1}{t^2})$, $||\vec{r}'(t)|| = 2t + \frac{1}{t}$, $\vec{r}'(t) \times \vec{r}''(t) = (-\frac{4}{t}, \frac{2}{t^2}, 4)$, and $||\vec{r}'(t) \times \vec{r}''(t)|| = \frac{2}{t^2} + 4$. Then

$$\kappa(t) = \frac{\frac{2}{t^2} + 4}{(2t + \frac{1}{t})^3} = \frac{2t + 4t^3}{(2t^2 + 1)^3} = \frac{2t}{(2t^2 + 1)^2}.$$

(c) The osculating plane is perpendicular to $\vec{T} \times \vec{N}$, so we may use $\vec{r}'(t) \times \vec{r}''(t)$ as the normal vector since scalar factors do not matter. At point P, we have $\vec{r}'(1) \times \vec{r}''(1) = (-4, 2, 4)$ and at point Q, we have $\vec{r}'(2) \times \vec{r}''(2) = (-2, \frac{1}{2}, 4)$. Then the osculating planes are:

$$\mathcal{P}_P : -4(x-2) + 2(y-1) + 4z = 0$$
$$\mathcal{P}_Q : -2(x-4) + \frac{1}{2}(y-4) + 4(z - \ln(2)) = 0.$$

To find the line of intersection of the two planes, we need the cross product of their normal vectors: $(-4, 2, 4) \times (-2, \frac{1}{2}, 4) = (6, 8, 2)$. If we set $x = 0$, then solving the system of equations $2(y-1) + 4z = 0$, $\frac{1}{2}(y-4) + 4(z - \ln(2)) = 0$ yields the point $(0, -\frac{8\ln(2)}{3}, \frac{1}{2} + \frac{4}{3}\ln(2))$ in the intersection of the two planes. Then we may write the line of intersection as

$$(6, 8, 2)t + (0, -\frac{8\ln(2)}{3}, \frac{1}{2} + \frac{4}{3}\ln(2)).$$

Problem 7.

(a) Recall that

$$x = \rho\cos\theta\sin\phi,$$
$$y = \rho\sin\theta\sin\phi,$$
$$z = \rho\cos\phi.$$

Using chain rule, we get

$$\begin{aligned}\frac{\partial f}{\partial \rho} &= 2x(\frac{\partial x}{\partial \rho}) - y(\frac{\partial x}{\partial \rho}) - x(\frac{\partial y}{\partial \rho}) + 3z^2(\frac{\partial z}{\partial \rho}) \\ &= 2\rho\cos^2\theta\sin^2\phi - 2\rho\cos\theta\sin\theta\sin^2\phi + 3\rho^2\cos^3\phi, \\ \frac{\partial f}{\partial \theta} &= -2\rho^2\cos\theta\sin\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi - \rho^2\cos^2\theta\sin^2\phi, \\ \frac{\partial f}{\partial \phi} &= 2\rho^2\cos^2\theta\sin\phi\cos\phi - 2\rho^2\sin\theta\cos\theta\sin\phi\cos\phi - 3\rho^3\cos^2\phi\sin\phi.\end{aligned}$$

(b) Implicitly differentiating $3 = x^2 - xy + z^3$ with respect to x, we have $0 = 2x - y - x\frac{\partial y}{\partial x} + 3z^2\frac{\partial z}{\partial x}$. At $(2, 1, 1)$, this simplifies to $0 = 4 - 1 - 2\frac{\partial y}{\partial x} + 3\frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x} = \frac{2}{3}\frac{\partial y}{\partial x} - 1$.

(c) The maximum rate of change occurs in the direction of the gradient $\nabla f(x, y, z) = (2x - y, -x, 3z^2)$ with rate $||\nabla f(x, y, z)|| = \sqrt{(2x-y)^2 + x^2 + 9z^4}$. At $(2, 1, 1)$, this is a rate of $\sqrt{9 + 4 + 9} = \sqrt{22}$ in the direction of $(3, -2, 3)$.

Problem 8.

(a) The quadratic equation yields

$$x = \frac{-(4) \pm \sqrt{16 - 4(1)(10)}}{2(1)} = \frac{-4 \pm \sqrt{-24}}{2},$$

both of which are not real because $\sqrt{-24}$ is not real.

(b) (a)

$$\frac{3-2i}{4+3i} = \frac{3-2i}{4+3i}\frac{4-3i}{4-3i} = \frac{6-17i}{25} = \frac{6}{25} - \frac{17}{25}i.$$

(b)

$$\begin{aligned} -2 + 2i^{\frac{1}{3}} &= -2 + 2(1^{\frac{1}{6}}) \\ &= -2 + 2e^{\frac{2\pi i}{6}} \\ &= -2 + 2(\sin(\frac{\pi}{3}) + \cos(\frac{\pi}{3})i) \\ &= -2 + 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\ &= -1 + \sqrt{3}i \end{aligned}$$

(c) Recall that $\cos(\theta) = \frac{e^{\theta i}+e^{-\theta i}}{2}$, so $\cos(3\theta) = \frac{e^{3\theta i}+e^{-3\theta i}}{2}$. But observe that $\cos(\theta)^3 = (\frac{e^{\theta i}+e^{-\theta i}}{2})^3 = \frac{e^{3\theta i}+3e^{\theta i}+3e^{-\theta i}+e^{-3\theta i}}{8}$. Then $\cos(3\theta) = 4\cos(\theta)^3 - 3\cos(\theta)$.

Problem 9.

(a) We have $f_x(x,y) = y\cos(xy)$ and $f_y(x,y) = x\cos(xy)$. Then $f_x(x,y) = 0$ if $y = 0$ or $xy = \frac{\pi}{2} + k\pi$ for any integer k. Similarly, $f_x(x,y) = 0$ if $x = 0$ or $xy = \frac{\pi}{2} + k\pi$ for any integer k. In order for both to be zero, we must have that $(x,y) = (0,0)$ or $xy = (k + \frac{1}{2})\pi$ for any integer k. We can alternatively characterize the critical points of f as $(0,0)$ and the points $(x, \frac{(k+\frac{1}{2})\pi}{x})$ for nonzero x and integers k.

(b) We have $f_{xx}(x,y) = -y^2\sin(xy), f_{xy}(x,y) = \cos(xy) - xy\sin(xy), f_{yy}(x,y) = -x^2\sin(xy)$. Then $D(x,y) = x^2y^2\sin^2(xy) - (\cos^2(xy) - 2xy\cos(xy)\sin(xy) + x^2y^2\sin^2(xy)) = 2xy\cos(xy)\sin(xy) - \cos^2(xy)$.

At $(0,0)$, $D(0,0) = -1$, so it is a saddle point of f. At $(x, \frac{(k+\frac{1}{2})\pi}{x})$ for an integer k, $D(x,y) = 0$ so the second partial derivative test is inconclusive. However, $f(x, \frac{(k+\frac{1}{2})\pi}{x}) = 1$ when k is even and $f(x, \frac{(k+\frac{1}{2})\pi}{x}) = -1$ when k is odd. Since $f(x,y) = \sin(xy)$ is bounded by -1 and 1, clearly f has local maxima when k is even and local minima when k is odd.

Problem 10.

Considering only points (x, y, z) satisfying $z = x^2 + 3y^2$, we have that

$$f(x, y, x^2 + 3y^2) = x^2 + 2x(x^2 + 3y^2) + \frac{y^2}{4} + (x^2 + 3y^2)^2 - 3$$
$$= (x + (x^2 + 3y^2))^2 + \frac{y^2}{4} - 3.$$

Observe that the non-constant terms are always non-negative and notice that $f(0, 0, 0) = -3$ and $f(-1, 0, 1) = -3$. Then -3 must be the absolute minimum value of f on the given elliptic paraboloid.

To see that these are the only points on the elliptic paraboloid where f attains the value of -3, observe that we need $(x + (x^2 + 3y^2))^2 = 0$ and $\frac{y^2}{4} = 0$. Then $y = 0$ from the latter condition, so we can simplify the former condition to $(x + x^2)^2 = 0$ which is equivalent to $x^2 + x = 0$. This occurs precisely when $x = 0$ or $x = -1$.

Alternatively, the problem can be solved using Lagrange multipliers, as explained during the review session.