PRACTICE FINAL SOLUTIONS

FALL 2018

UN1201: CALCULUS III

Problem 1.

(a) True, \( \vec{u} \cdot (2\vec{v} - \vec{w}) = 2\vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w} = 2(1) - (2) = 0. \)

(b) True, reparametrize the curve \( \vec{r}(t) = (x, f(x)) \) according to arc length so \( 0 = \kappa = |\vec{r}''(s)| \). Then \( \vec{r}''(s) = 0 \), so \( r'(s) \) is constant. Then \( \vec{r}(s) = r_0 + su \) for some vector \( r_0 \) and unit vector \( \vec{u} \).

(c) False: if \( S \) is a level surface of a function \( g \), then the gradient of \( f \) is parallel to the gradient of \( g \) at the point \( P \). Thus the gradient of \( f \) is perpendicular, not parallel, to the tangent plane at \( P \).

(d) True, observe that \( x^4 + y^4 - x^2y^2 = (x^4 - 2x^2y^2 + y^4) + x^2y^2 = (x^2 - y^2)^2 + (xy)^2 \geq 0 \) for all real \( (x, y) \).

Problem 2.

(a) Let \( (x_0, y_0, z_0) \) be the intersection point of \( \ell \) and \( m \). From \( \frac{z_0 - z}{c} + 3 = x_0 = \frac{1 - z_0}{3} + 2 \), we deduce \( 3z_0 - 6 + 3c = c - cz_0 \) and so \((3 + c)z_0 + (-6 + 2c) = 0 \). Then \( z_0 = \frac{6 - 2c}{3 + c} \).

From \( 3\frac{z_0 - z}{c} - 4 = y_0 = -2\frac{1 - z_0}{c} + 3 \), we deduce \( 9z_0 - 18 = -2c + 2cz_0 + 21c \) and so \((9 - 2c)z_0 + (-18 - 19c) = 0 \). Then \( \frac{6 - 2c}{3 + c} = z_0 = \frac{18 + 19c}{9 - 2c} \), so

\[
0 = (6 - 2c)(9 - 2c) - (18 + 19c)(3 + c) = 4c^2 - 30c + 54 - 19c^2 - 75c - 54 = -15c^2 - 105c = -15c(c + 7).
\]

Being the denominator of \( \frac{z_0 - 2}{c} \), we know that \( c \) must be nonzero. Then we must have \( c = -7 \).

(b) Let \( t = x - 2 = \frac{3 - y}{2} = \frac{1 - z}{3} \), so \( \ell \) is given by \( x = t + 2, y = -2t + 3, z = -3t + 1 \), i.e. \((1, -2, -3)t + (2, 3, 1)\). Similarly, letting \( s = x - 3 = \frac{y + 4}{3} = \frac{z - 2}{3} \), we have that \( m \) is given by \((1, 3, -7)t + (3, -4, 2)\).

Then the plane containing both lines is perpendicular to \((1, -2, -3) \times (1, 3, -7) = (23, 4, 5)\). From our expressions in (a), we know that \( z_0 = \frac{6 - 2(1 - 7)}{3 + (1 - 7)} = \frac{20}{4} = -5 \). Then \( x_0 = \frac{1 - z_0}{3} + 2 = \frac{1 - (-5)}{3} + 2 = 4 \) and \( y_0 = -2(x_0 - 2) + 3 = -2(4 - 2) + 3 = -1 \).

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Hence \((4, -1, -5)\) is the point of intersection of \(l\) and \(m\) (check that this is true by verifying that both symmetric equations hold for this point!).

The plane containing both lines can be written as

\[
23(x - 4) + 4(y + 1) + 5(z + 5) = 0,
\]

or as

\[
23x + 4y + 5z - 63 = 0.
\]

**Problem 3.**

(a) A sketch should look like:

(b) The first partials are \(f_x(x, y) = 3x^2 - y - 1\) and \(f_y(x, y) = -x + 2y\). If \(0 = f_y(x, y) = -x + 2y\), then \(x = 2y\), so \(0 = f_x(x, y) = 3(2y)^2 - y - 1 = 12y^2 - y - 1\) which has solutions \(-\frac{1}{4}\) and \(\frac{1}{3}\). Since \(x = 2y\), the two critical points of \(f\) are \((-\frac{1}{2}, -\frac{1}{4})\) and \((\frac{2}{3}, \frac{1}{3})\). We see that \((\frac{2}{3}, \frac{1}{3})\) lies in the region \(R\) since \(\frac{1}{3} \geq 0\), \(\frac{2}{3} \geq 0\), and \(\frac{1}{3} + \frac{2}{3} = 1 \leq 2\). This is the only critical point of \(f\) in \(R\), since \(-\frac{1}{2} < 0\).

(c) The second partials are \(f_{xx}(x, y) = 6x\), \(f_{xy}(x, y) = -1\), \(f_{yy}(x, y) = 2\). At the critical point \((\frac{2}{3}, \frac{1}{3})\), we have that \(D = (6\frac{2}{3})(2) - (-1)^2 = 8 - 1 = 7 > 0\) with \(f_{xx}(\frac{2}{3}, \frac{1}{3}) = 4 > 0\) so it is a local minimum for \(f\).

We must also check \(f\) along the boundary of \(R\), i.e. where \(x = 0\), \(y = 0\), or \(x + y = 2\). When \(x = 0\), we have \(f(0, y) = y^2\) which clearly has a minimum of 0 and a maximum of 4 when restricted to \(x = 0\) and \(R\). When \(y = 0\), we have \(f(x, 0) = x^3 - x\) which has a minimum of \(-\frac{2\sqrt{3}}{9}\) (differentiate with respect to \(x\) to find the single-variable minimum at \(x = -\sqrt{\frac{2}{3}}\)) and a maximum of 6 (at \(x = 2\)) along \(y = 0\) in \(R\). When \(x + y = 2\), we have \(y = 2 - x\). Then \(f(x, 2 - x) = x^3 - x(2-x) + (2-x)^2 - x = x^3 + 2x^2 - 7x + 4\). Using single-variable techniques, we have that \(f\) has a minimum of 0 (at \(x = 1\)) and a maximum of 6 (at \(x = 2\)) along \(y = 2 - x\) in \(R\).
The only critical point of \( f \) in \( \mathbb{R} \) is \((\frac{2}{3}, \frac{1}{3})\) with local minimum \( f(\frac{2}{3}, \frac{1}{3}) = -\frac{13}{27} \), which is less than the minimum value of \(-\frac{2\sqrt{3}}{9}\) along the boundary of \( \mathbb{R} \), so \( f \) attains its absolute minimum in \( \mathbb{R} \) of \(-\frac{13}{27}\) at \((\frac{2}{3}, \frac{1}{3})\). Since there is no local maximum of \( f \) in the interior of \( \mathbb{R} \), the absolute maximum of \( f \) is its maximum along the boundary, which is the value of 6 attained at \((2,0)\).

**Problem 4.**

(a) The level surfaces are of the form \( k - 14 = x^2 - 2y^2 + \frac{z^2}{9} \). This is a hyperboloid of one sheet when \( k - 14 > 0 \), i.e. when \( k > 14 \), and a hyperboloid of two sheets when \( k - 14 < 0 \), i.e. when \( k < 14 \). When \( k = 14 \), we have an elliptic cone.

(b) The given level surface is \( F(x,y,z) := x^2 - 2y^2 + \frac{z^2}{9} - 2 = 0 \). Then \( F_x(x,y,z) = 2x, F_y(x,y,z) = -4y, F_z(x,y,z) = \frac{2z}{9} \). Then the tangent plane to the given level surface at \((1,2,9)\) is given by \( F_x(1,2,9)(x-1) + F_y(1,2,9)(y-2) + F_z(1,2,9)(z-9) = 0 \), so it is
\[
2(x-1) - 8(y-2) + 2(z-9) = 0,
\]
which can also be written as
\[
2x - 8y + 2z - 4 = 0.
\]

**Problem 5.**

(a) Along \((y = 0)\), we have \( \lim_{(x,z) \to (0,0)} f(x,0,z) = \lim_{(x,z) \to (0,0)} \frac{x^2 + z^2}{x + z^2} = 1 \). However, along \((x = 0, z = 0)\), we have \( \lim_{y \to 0} f(0,y,0) = \lim_{y \to 0} \frac{-y^2}{y} = -1 \). Hence, the limit \( \lim_{(x,y,z) \to (0,0,0)} f(x,y,z) \) does not exist.

Switching to spherical coordinates, we have that
\[
\lim_{(x,y,z) \to (0,0,0)} g(x,y,z) = \lim_{\rho \to 0} g(\rho, \theta, \phi) = \lim_{\rho \to 0} \frac{\rho^4 \cos^4 \theta \sin^4 \phi + \rho^4 \sin^4 \theta \sin^4 \phi + \rho^4 \cos^4 \phi}{\rho^4} \]
\[
= \lim_{\rho \to 0} \rho^2(\cos^4 \theta \sin^4 \phi + \sin^4 \theta \sin^4 \phi + \cos^4 \phi) \]
\[
= 0.
\]

(b) In order to be sure that the bridge will not collapse, we must have that the absolute maximum of \( w \) in the region given by \( 14.86 \leq x \leq 15.14, \ 3.86 \leq y \leq 4.14, \) and \( 2.86 \leq z \leq 3.14 \) is less than 22.5. However, we can observe that \( w \) always increases with respect to \( x, y, \) and \( z \). Thus, we only need to ensure that \( w(15.14,4.14,3.14) > 22.5 \).
Without directly calculating $w(15.14, 4.14, 3.14)$, we can try a linear approximation using the tangent plane at $(15, 4, 3)$ to overestimate $w(15.14, 4.14, 3.14)$. Calculate that $w(15, 4, 3) = 21, w_x(15, 4, 3) = \frac{33}{35}, w_y(15, 4, 3) = \frac{99}{28},$ and $w_z(15, 4, 3) = \frac{15}{7}$. Then the linear approximation is

$$w(15.14, 4.14, 3.14) \approx w(15, 4, 3) + w_x(15, 4, 3)(0.14) + w_y(15, 4, 3)(0.14) + w_z(15, 4, 3)(0.14)$$

$$= 21 + \left(\frac{33}{35} + \frac{99}{28} + \frac{15}{7}\right)(0.14)$$

$$\leq 21 + (1 + 4 + 3)(0.14)$$

$$= 21 + 8\left(\frac{7}{50}\right)$$

$$= 21 + \frac{56}{50}$$

$$= 22.12$$

$$< 22.5.$$

However, we do not know if the linear approximation overestimates or underestimates the value of $w(15, 4, 3)$, so we cannot be absolutely sure.

**Problem 6.**

(a) Observe that $\vec{r}(1) = (2, 1, 0) = P$ and $\vec{r}(2) = (4, 4, \ln(2)) = Q$. Using the components of $\vec{r}'(t) = (2, 2t, \frac{1}{t})$, the arc length of $C$ between $P$ and $Q$ is

$$L = \int_1^2 \sqrt{4 + 4t^2 + \frac{1}{t^2}}\,dt$$

$$= \int_1^2 \sqrt{(2t + \frac{1}{t})^2}\,dt$$

$$= \int_1^2 (2t + \frac{1}{t})\,dt$$

$$= t^2 + \ln(t)\big|_1^2$$

$$= (4 - 1) + (\ln(2) - 0)$$

$$= 3 + \ln(2).$$

(b) Compute that $\vec{r}''(t) = (0, 2, -\frac{1}{t^2})$, $||\vec{r}'(t)|| = 2t + \frac{1}{t}$, $\vec{r}'(t) \times \vec{r}''(t) = (-\frac{4}{t}, \frac{2}{t^2}, 4)$, and $||\vec{r}'(t) \times \vec{r}''(t)|| = \frac{2}{t^2} + 4$. Then

$$\kappa(t) = \frac{\frac{2}{t^2} + 4}{(2t + \frac{1}{t})^3} = \frac{2t + 4t^3}{(2t^2 + 1)^3} = \frac{2t}{(2t^2 + 1)^2}.$$
(c) The osculating plane is perpendicular to $\vec{T} \times \vec{N}$, so we may use $\vec{r}'(t) \times \vec{r}''(t)$ as the normal vector since scalar factors do not matter. At point $P$, we have $\vec{r}'(1) \times \vec{r}''(1) = (-4, 2, 4)$ and at point $Q$, we have $\vec{r}'(2) \times \vec{r}''(2) = (-2, \frac{1}{2}, 4)$. Then the osculating planes are:

\[ P_P : -4(x - 2) + 2(y - 1) + 4z = 0 \]
\[ P_Q : -2(x - 4) + \frac{1}{2}(y - 4) + 4(z - \ln(2)) = 0. \]

To find the line of intersection of the two planes, we need the cross product of their normal vectors:

\[ (−4,2,4) \times (−2,\frac{1}{2},4) = (6,8,2) \]

If we set $x = 0$, then solving the system of equations

\[ 2(y - 1) + 4z = 0, \]
\[ \frac{1}{2}(y - 4) + 4(z - \ln(2)) = 0 \]

yields the point

\[ (0,−\frac{8 \ln(2)}{3},\frac{1}{2} + \frac{4}{3} \ln(2)) \]

in the intersection of the two planes. Then we may write the line of intersection as

\[ (6,8,2)t + (0,−\frac{8 \ln(2)}{3},\frac{1}{2} + \frac{4}{3} \ln(2)) \]

Problem 7.

(a) Recall that

\[ x = \rho \cos \theta \sin \phi, \]
\[ y = \rho \sin \theta \sin \phi, \]
\[ z = \rho \cos \phi. \]

Using chain rule, we get

\[ \frac{\partial f}{\partial \rho} = 2x(\frac{\partial x}{\partial \rho}) - y(\frac{\partial x}{\partial \rho}) - x(\frac{\partial y}{\partial \rho}) + 3z^2(\frac{\partial z}{\partial \rho}) \]
\[ = 2\rho \cos^2 \theta \sin^2 \phi - 2\rho \cos \theta \sin \theta \sin^2 \phi + 3\rho^2 \cos^3 \phi, \]
\[ \frac{\partial f}{\partial \theta} = -2\rho^2 \cos \theta \sin \theta \sin^2 \phi + \rho^2 \sin^2 \theta \sin^2 \phi - \rho^2 \cos^2 \theta \sin^2 \phi, \]
\[ \frac{\partial f}{\partial \phi} = 2\rho^2 \cos^2 \theta \sin \phi \cos \phi - 2\rho^2 \sin \theta \cos \theta \sin \phi \sin \phi - 3\rho^3 \cos^2 \phi \sin \phi. \]

(b) Implicitly differentiating $3 = x^2 - xy + z^3$ with respect to $x$, we have $0 = 2x - y - x\frac{\partial y}{\partial x} + 3z^2 \frac{\partial z}{\partial x}$. At $(2,1,1)$, this simplifies to $0 = 4 - 1 - 2\frac{\partial y}{\partial x} + 3\frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x} = \frac{2}{3} \frac{\partial y}{\partial x} - 1$.

(c) The maximum rate of change occurs in the direction of the gradient $\nabla f(x, y, z) = (2x - y, -x, 3z^2)$ with rate $||\nabla f(x, y, z)|| = \sqrt{(2x - y)^2 + x^2 + 9z^4}$. At $(2,1,1)$, this is a rate of $\sqrt{7 + 4 + 7} = \sqrt{18}$ in the direction of $(3, -2, 3)$.

Problem 8.
(a) The quadratic equation yields
\[ x = \frac{-(4) \pm \sqrt{16 - 4(1)(10)}}{2(1)} = \frac{-4 \pm \sqrt{-24}}{2}, \]
both of which are not real because \( \sqrt{-24} \) is not real.

(b) (a) \[ 3 - 2i \quad = \quad 3 - 2i \quad 4 - 3i \quad = \quad 6 - 17i \quad = \quad \frac{6}{25} \quad - \quad \frac{17}{25}i. \]

(b) \[ -2 + 2i \theta = -2 + 2(1 + i) \]
\[ = -2 + 2e^{\frac{i\pi}{2}} \]
\[ = -2 + 2\left(\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)i\right) \]
\[ = -2 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \]
\[ = -1 + \sqrt{3}i. \]

(c) Recall that \( \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \), so \( \cos(3\theta) = \frac{e^{i3\theta} + e^{-i3\theta}}{2} \). But observe that \( \cos(\theta)^3 = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^3 = \frac{e^{i3\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}}{8} \). Then \( \cos(3\theta) = 4\cos(\theta)^3 - 3\cos(\theta). \)

Problem 9.

(a) We have \( f_x(x, y) = y \cos(xy) \) and \( f_y(x, y) = x \cos(xy) \). Then \( f_x(x, y) = 0 \) if \( y = 0 \) or \( xy = \frac{\pi}{2} + k\pi \) for any integer \( k \). Similarly, \( f_x(x, y) = 0 \) if \( x = 0 \) or \( xy = \frac{\pi}{2} + k\pi \) for any integer \( k \). In order for both to be zero, we must have that \( (x, y) = (0, 0) \) or \( xy = (k + \frac{1}{2})\pi \) for any integer \( k \). We can alternatively characterize the critical points of \( f \) at \( (0, 0) \) and the points \( (x, \frac{(k + \frac{1}{2})\pi}{x}) \) for nonzero \( x \) and integers \( k \).

(b) We have \( f_{xx}(x, y) = -y^2 \sin(xy), f_{xy}(x, y) = x \cos(xy) - xy \sin(xy), f_{yy}(x, y) = -x^2 \sin(xy) \). Then \( D(x, y) = x^2y^2 \sin^2(xy) - (\cos^2(xy) - 2xy \cos(xy) \sin(xy) + x^2y^2 \sin^2(xy)) = 2xy \cos(xy) \sin(xy) - \cos^2(xy) \).

At \( (0, 0), D(0, 0) = -1 \), so it is a saddle point of \( f \). At \( (x, \frac{(k + \frac{1}{2})\pi}{x}) \) for an integer \( k \), \( D(x, y) = 0 \) so the second partial derivative test is inconclusive. However, \( f(x, \frac{(k + \frac{1}{2})\pi}{x}) = 1 \) when \( k \) is even and \( f(x, \frac{(k + \frac{1}{2})\pi}{x}) = -1 \) when \( k \) is odd. Since \( f(x, y) = \sin(xy) \) is bounded by \(-1\) and \( 1 \), clearly \( f \) has local maxima when \( k \) is even and local minima when \( k \) is odd.

Problem 10.
Considering only points \((x, y, z)\) satisfying \(z = x^2 + 3y^2\), we have that

\[
f(x, y, x^2 + 3y^2) = x^2 + 2x(x^2 + 3y^2) + \frac{y^2}{4} + (x^2 + 3y^2)^2 - 3
\]

\[
= (x + (x^2 + 3y^2))^2 + \frac{y^2}{4} - 3.
\]

Observe that the non-constant terms are always non-negative and notice that 
\(f(0, 0, 0) = -3\) and 
\(f(-1, 0, 1) = -3\). Then \(-3\) must be the absolute minimum value of \(f\) on the given elliptic paraboloid.

To see that these are the only points on the elliptic paraboloid where \(f\) attains the value of \(-3\), observe that we need 
\((x + (x^2 + 3y^2))^2 = 0\) and 
\(\frac{y^2}{4} = 0\). Then \(y = 0\) from the latter condition, so we can simplify the former condition to 
\((x + x^2)^2 = 0\) which is equivalent to \(x^2 + x = 0\). This occurs precisely when \(x = 0\) or \(x = -1\).

Alternatively, the problem can be solved using Lagrange multipliers, as explained during the review session.