Calculus III: Midterm Exam 2

November 13, 2018

1. (a) (i) True. The function has the form $z = ax + by$, which is equivalent to

$$ax + by - z = 0.$$

One can easily check that the all the second partial derivatives equal 0.

(ii) False. The point $(1,1)$ (corresponding to $t = 0$) should be on the graph, but it is not.

(b) (i) One example is $f(x, y) = x + |y|$. We have $\frac{\partial f}{\partial x} = 1$ is continuous for all (x, y), but $\frac{\partial f}{\partial y}$ does not exist at $(x, y) = (0, 0)$, because $|y|$ is not differentiable at $y = 0$.

(ii) Let

$$\vec{r}(t) = \langle t, t^3, 0 \rangle$$
$$\vec{r}'(t) = \langle 1, 3t^2, 0 \rangle$$
$$\vec{r}''(t) = \langle 0, 6t, 0 \rangle$$

$$\kappa(t) = \frac{|6t|}{(\sqrt{1 + 9t^4})^3}$$

We get $\kappa(t) = 0 \iff t = 0$, as desired.

2. (a)

$$\vec{r}(t) = \langle \sin^2 t, \sin^3 t, \cos^3 t \rangle$$

$$\vec{r}'(t) = \langle 2\sin t\cos t, 3\sin^2 t\cos t, -3\cos^2 t\sin t \rangle$$

$$|\vec{r}'(t)| = \sqrt{4\sin^2 t\cos^2 t + 9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t} = \sqrt{4\sin^2 t\cos^2 t + 9\sin^2 t\cos^2 t} =$$

$$= \sqrt{13}|\sin t\cos t| = \sqrt{13}\sin t\cos t$$

The unit tangent vector is:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle \frac{2}{\sqrt{13}}, \frac{3\sin t}{\sqrt{13}}, -\frac{3\cos t}{\sqrt{13}} \rangle$$

The unit normal vector is:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$\vec{T}'(t) = \langle 0, \frac{3\cos t}{\sqrt{13}}, \frac{3\sin t}{\sqrt{13}} \rangle$$

$$|\vec{T}'(t)| = \sqrt{\frac{9}{13}\cos^2 t + \frac{9}{13}\sin^2 t} = \frac{3}{\sqrt{13}}$$

$$\vec{N}(t) = \langle 0, \cos t, \sin t \rangle$$

The binormal vector is:

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \langle \frac{3}{\sqrt{13}}, -\frac{2\sin t}{\sqrt{13}}, \frac{2\cos t}{\sqrt{13}} \rangle$$

(b) The length of the curve is

$$\int_0^{\pi/2} |\vec{r}'(t)|dt = \int_0^{\pi/2} \sqrt{13}\sin t\cos t dt = \frac{\sqrt{13}}{2}\int_0^{\pi/2} \sin(2t)dt = \frac{\sqrt{13}}{4}\int_0^{\pi} \sin(u)du =$$

$$= \frac{\sqrt{13}}{4}(-\cos(\pi) + \cos 0) = \frac{\sqrt{13}}{2}.$$

3. (a) We have $z = \ln(x) = \ln(\frac{y^2}{4})$. Moreover, $x \in [\frac{1}{2}, 2]$ means $\frac{y^2}{4} \in [\frac{1}{2}, 2]$, so $y^2 \in [2, 8]$, so $y \in [-2\sqrt{2}, \sqrt{2}] \cup [\sqrt{2}, 2\sqrt{2}]$. Let:

$$y = t$$

$$x = \frac{t^2}{4}$$

$$z = \ln(\frac{t^2}{4})$$

This way, the curve is parametrized as follows:

$$\vec{r}(t) = \langle \frac{t^2}{4}, t, \ln(\frac{t^2}{4}) \rangle.$$

The curvature of the curve is given by

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

$$\vec{r}'(t) = \langle \frac{t}{2}, 1, \frac{2}{t} \rangle$$

$$\vec{r}''(t) = \langle \frac{1}{2}, 0, -\frac{2}{t^2} \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle -\frac{2}{t^2}, \frac{2}{t}, -\frac{1}{2} \rangle$$

Putting everything together, we obtain the curvature:

$$\kappa(t) = \frac{(\frac{4}{t^4} + \frac{4}{t^2} + \frac{1}{4})^{1/2}}{(\frac{t^2}{4} + 1 + \frac{4}{t^2})^{3/2}}$$

(b) From the formula

$$s(t) = \int_0^t |r'(u)| du,$$

we write t as a function of the arc length: $t = t(s)$. Then we reparametrize as follows:

$$\vec{r}(t) = \vec{r}(t(s)) = \langle \frac{t(s)^2}{4}, t(s), \ln(\frac{t(s)^2}{4}) \rangle.$$

(c)

$$\vec{r}(t) = \langle t, t\cos t, t\sin t \rangle$$

The osculating plane of a curve C at a point P is determined by the vectors $\vec{T}$ and $\vec{N}$.

$$\vec{r}'(t) = \langle 1, \cos t - t\sin t, \sin t + t\cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{1+1+t^2} = \sqrt{2+t^2}$$

$$\vec{T}(t) = \langle \frac{1}{\sqrt{2+t^2}}, \frac{\cos t - t\sin t}{\sqrt{2+t^2}}, \frac{\sin t + t\cos t}{\sqrt{2+t^2}} \rangle$$

$$\vec{T}'(t) = \langle -\frac{1}{2}(2+t^2)^{-3/2}(2t), -\frac{1}{2}(2+t^2)^{-3/2}(2t)(\cos t - t\sin t) - (2+t^2)^{-1/2}(2\sin t + t\cos t),$$
$$-\frac{1}{2}(2+t^2)^{-3/2}(2t)(\sin t + t\cos t) + (2+t^2)^{-1/2}(2\cos t - t\sin t)\rangle$$

At $P = (\pi, \pi, 0)$, where $t = \pi$, we have:

$$\vec{T}(\pi) = \frac{1}{\sqrt{1+\pi^2}}\langle 1, -1, -\pi\rangle$$

$$\vec{T}'(\pi) = \frac{1}{(1+\pi^2)^{3/2}}\langle -\pi, 3\pi+\pi^3, -\pi^2-4\rangle$$

A normal vector to the osculating plane is $\vec{T}(\pi) \times \vec{N}(\pi) = \vec{T}(\pi) \times \frac{\vec{T}'(\pi)}{|\vec{T}'(\pi)|}$. Since scalars don't change this property, it follows that $\vec{T}(\pi) \times \vec{T}'(\pi)$ is also a normal vector to the osculating plane. Ignoring scalars once again, we obtain a normal vector:

$$\langle 1, -1, -\pi\rangle \times \langle \pi, 3\pi+\pi^3, -\pi^2-4\rangle = \langle (\pi^2+2)^2, 2(\pi^2+2), \pi(\pi^2+2)\rangle = (\pi^2+2)\langle \pi^2+2, 2, \pi\rangle.$$

Ignoring scalars, we can use $\langle \pi^2+2, 2, \pi\rangle$ as a normal vector, and we get an equation of the osculating plane:

$$(\pi^2+2)(x-\pi) + 2(y-\pi) + \pi z = 0.$$

4. (a) (i) This limit does not exist:

$$\lim_{(x,x)\to(0,0)} \frac{x^3\cos x}{x^2+x^4} = \lim_{x\to 0} \frac{x}{1+x^2} = 0$$

$$\lim_{(y^2,y)\to(0,0)} \frac{y^4\cos y}{y^4+y^4} = \frac{1}{2}$$

Since the limits above don't coincide, the overall limit does not exist.

(ii) First, note that we have

$$\lim_{z\to 0} \frac{\cos z - 1}{z^2} = \lim_{z\to 0} \frac{-\sin z}{2z} = \lim_{z\to 0} \frac{-\cos z}{2} = \frac{-1}{2}.$$

Moreover,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+1}{x^2-y^2+2} = \frac{1}{2}.$$

Therefore, the limit exists, and $\lim_{(x,y,z)\to(0,0,0)} \frac{(x^2+y^2+1)(\cos z-1)}{(x^2-y^2+2)(z^2)} = -\frac{1}{4}$.

(iii) If $p=1$, we need to show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$ does not exists. That is because $\lim_{(x,x)\to(0,0)} \frac{x^2}{2x^4} = \lim_{x\to 0} \frac{1}{x^2} = +\infty$, but $\lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$.
If $p=2$, we need to show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$ does not exist. That is because $\lim_{(x,x)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2}$, but $\lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$.
If $p > 3$, then $(xy)^p$ is has degree $2p > 4$, so the limit exists and $\lim_{(x,y)\to(0,0)} \frac{(xy)^p}{x^4+y^4} = 0$.

(b) A level surface has the form $f(x,y,z) = k$, so

$$x + y^2 + z^2 = k.$$

For any k, as above, the level surface is an elliptic paraboloid.

(c)

$$g(x,y,z) = (x+y^2+z^2)e^{y^2+z}$$

$$\frac{\partial g}{\partial x} = 1\cdot e^{y^2+z} + (x+y^2+z^2)\cdot 0 = e^{y^2+z}$$

$$\frac{\partial g}{\partial y} = (2y)\cdot e^{y^2+z} + (x+y^2+z^2)\cdot e^{y^2+z}\cdot(2y) = 2y(x+y^2+z^2+1)e^{y^2+z}$$

$$\frac{\partial g}{\partial z} = (2z)\cdot e^{y^2+z} + (x+y^2+z^2)e^{y^2+z}\cdot 1 = e^{y^2+z}(x+y^2+z^2+2z)$$

$$\frac{\partial^2 g}{\partial x \partial z} = e^{y^2+z}\cdot 1 = e^{y^2+z}$$

5. (a)

$$f(x, y, z) = x^2 - 2xy + 7z^2 - 15$$
$$\nabla f(x, y, z) = \langle 2x - 2y, -2x, 14z \rangle$$
$$\nabla f(4, 1, -1) = \langle 8 - 2, -8, -14 \rangle = \langle 6, -8, -14 \rangle$$

This means that the vector $\langle 6, -8, -14 \rangle$ is the normal to the tangent plane, so an equation for the tangent plane is:

$$6(x - 4) - 8(y - 1) - 14(z + 1) = 0$$

$$3x - 4y - 7z = 15$$

(b) The line perpendicular to the tangent plane has direction $\langle 6, -8, -14 \rangle$, so a parametric equation for it is:

$$(4, 1, -1) + t(6, -8, -14) = (6t + 4, -8t + 1, -14t - 1).$$

This gives the corresponding symmetric equations:

$$\frac{x - 4}{6} = \frac{y - 1}{-8} = \frac{z + 1}{-14}.$$

(c) For this part, we need to estimate the possible error in the volume measurement:

$$V = 4(x - \frac{1}{2})yz + \frac{2}{3}y^2z^2$$

$$dV = 4(x - \frac{1}{2})ydz + 4(x - \frac{1}{2})zdy + 4yzdx + \frac{4}{3}yz^2dy + \frac{4}{3}y^2zdz$$

Since we have a possible error of .01 meter in each direction, the possible error for the volume is:

$$dV = 4 \cdot 3.5 \cdot 10 \cdot 0.01 + 4 \cdot 3.5 \cdot 3 \cdot 0.01 + 4 \cdot 10 \cdot 3 \cdot 0.01 + \frac{4}{3} \cdot 10 \cdot 3^2 \cdot 0.01 + \frac{4}{3} \cdot 10^2 \cdot 3 \cdot 0.01$$

$$dV = 8.82$$

When $(x, y, z) = (4, 10, 3)$, we get $V = 1020$. Even with a maximum error of 8.22, the volume would still be greater than 600, which means the submarine can accomodate all 6 people.