Calculus III: Midterm Exam 2
November 13, 2018

1. (a) (i) True. The function has the form $z = ax + by$, which is equivalent to

$$ax + by - z = 0.$$ 

One can easily check that all the second partial derivatives equal 0.

(ii) False. The point $(1, 1)$ (corresponding to $t = 0$) should be on the graph, but it is not.

(b) (i) One example is $f(x, y) = x + |y|$. We have $\frac{\partial f}{\partial x} = 1$ is continuous for all $(x, y)$, but $\frac{\partial f}{\partial y}$ does not exist at $(x, y) = (0, 0)$, because $|y|$ is not differentiable at $y = 0$.

(ii) Let

$$\vec{r}(t) = \langle t, t^3, 0 \rangle$$

$$\vec{r}'(t) = \langle 1, 3t^2, 0 \rangle$$

$$\vec{r}''(t) = \langle 0, 6t, 0 \rangle$$

$$\kappa(t) = \frac{|6t|}{(\sqrt{1 + 9t^4})^3}$$

We get $\kappa(t) = 0 \iff t = 0$, as desired.
2. (a) 
\[
\vec{r}(t) = (\sin^2 t, \sin^3 t, \cos^3 t)
\]
\[
\vec{r}'(t) = (2 \sin t \cos t, 3 \sin^2 t \cos t, -3 \cos^2 t \sin t)
\]
\[
|\vec{r}'(t)| = \sqrt{4 \sin^2 t \cos^2 t + 9 \sin^4 t \cos^2 t + 9 \cos^4 t \sin^2 t} = \sqrt{4 \sin^2 t \cos^2 t + 9 \sin^2 t \cos^2 t} = \sqrt{13} |\sin t \cos t| = \sqrt{13} \sin t \cos t
\]

The unit tangent vector is:
\[
\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left( \frac{2}{\sqrt{13}}, \frac{3 \sin t}{\sqrt{13}}, -\frac{3 \cos t}{\sqrt{13}} \right)
\]
The unit normal vector is:
\[
\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \left( 0, \frac{3 \cos t}{\sqrt{13}}, \frac{3 \sin t}{\sqrt{13}} \right)
\]
\[
|\vec{T}'(t)| = \sqrt{\frac{9}{13} \cos^2 t + \frac{9}{13} \sin^2 t} = \frac{3}{\sqrt{13}}
\]
\[
\vec{N}(t) = (0, \cos t, \sin t)
\]
The binormal vector is:
\[
\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \left( \frac{3}{\sqrt{13}}, -\frac{2 \sin t}{\sqrt{13}}, \frac{2 \cos t}{\sqrt{13}} \right)
\]

(b) The length of the curve is
\[
\int_0^{\pi/2} |\vec{r}'(t)| \, dt = \int_0^{\pi/2} \sqrt{13} \sin t \cos t \, dt = \frac{\sqrt{13}}{2} \int_0^{\pi/2} \sin(2t) \, dt = \frac{\sqrt{13}}{4} \int_0^\pi \sin(u) \, du = \frac{\sqrt{13}}{4} (-\cos(\pi) + \cos 0) = \frac{\sqrt{13}}{2}.
\]
3. (a) We have \( z = \ln(x) = \ln(\frac{t^2}{4}) \). Moreover, \( x \in \left[\frac{1}{2}, 2\right] \) means \( \frac{t^2}{4} \in \left[\frac{1}{2}, 2\right] \), so \( y^2 \in [2, 8] \), so \( y \in [-2\sqrt{2}, \sqrt{2}] \cup [\sqrt{2}, 2\sqrt{2}] \). Let:

\[
\begin{align*}
y &= t \\
x &= \frac{t^2}{4} \\
z &= \ln(\frac{t^2}{4})
\end{align*}
\]

This way, the curve is parametrized as follows:

\[
\vec{r}(t) = \left\langle \frac{t^2}{4}, t, \ln(\frac{t^2}{4}) \right\rangle.
\]

The curvature of the curve is given by

\[
\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}
\]

\[
\vec{r}'(t) = \left\langle \frac{t}{2}, 1, \frac{2}{t} \right\rangle
\]

\[
\vec{r}''(t) = \left\langle 0, -\frac{1}{t^2}, -\frac{2}{t^3} \right\rangle
\]

\[
\vec{r}'(t) \times \vec{r}''(t) = \left\langle -\frac{2}{t^2}, \frac{2}{t}, -\frac{1}{2} \right\rangle
\]

Putting everything together, we obtain the curvature:

\[
\kappa(t) = \frac{\left(\frac{4}{t} + \frac{4}{t^2} + \frac{1}{4}\right)^{1/2}}{\left(\frac{t^2}{4} + 1 + \frac{4}{t^2}\right)^{3/2}}
\]

(b) From the formula

\[
s(t) = \int_0^t |\vec{r}'(u)| \, du,
\]

we write \( t \) as a function of the arc length: \( t = t(s) \). Then we reparametrize as follows:

\[
\vec{r}(t) = \vec{r}(t(s)) = \left\langle \frac{t(s)^2}{4}, t(s), \ln(\frac{t(s)^2}{4}) \right\rangle.
\]

(c) \( \vec{r}'(t) = \langle t, t \cos t, t \sin t \rangle \)

The osculating plane of a curve \( C \) at a point \( P \) is determined by the vectors \( \vec{T} \) and \( \vec{N} \).

\[
\vec{r}'(t) = \langle 1, \cos t - t \sin t, \sin t + t \cos t \rangle
\]
\[ |\vec{r}'(t)| = \sqrt{1 + 1 + t^2} = \sqrt{2 + t^2} \]
\[ \vec{T}(t) = \left( \frac{1}{\sqrt{2 + t^2}}, \frac{\cos t - t \sin t}{\sqrt{2 + t^2}}, \frac{\sin t + t \cos t}{\sqrt{2 + t^2}} \right) \]
\[ \vec{T}'(t) = \left( -\frac{1}{2} (2 + t^2)^{-3/2} (2t), -\frac{1}{2} (2 + t^2)^{-3/2} (2t)(\cos t - t \sin t) - (2 + t^2)^{-1/2} (2 \sin t + t \cos t), \right. \]
\[ \left. -\frac{1}{2} (2 + t^2)^{-3/2} (2t)(\sin t + t \cos t) + (2 + t^2)^{-1/2} (2 \cos t - t \sin t) \right) \]

At \( P = (\pi, \pi, 0) \), where \( t = \pi \), we have:
\[ \vec{T}(\pi) = \frac{1}{\sqrt{1 + \pi^2}} (1, -1, -\pi) \]
\[ \vec{T}'(\pi) = \frac{1}{(1 + \pi^2)^{3/2}} (-\pi, 3\pi + \pi^3, -\pi^2 - 4) \]

A normal vector to the osculating plane is \( \vec{T}(\pi) \times \vec{N}(\pi) = \vec{T}(\pi) \times \frac{\vec{T}'(\pi)}{|\vec{T}'(\pi)|} \). Since scalars don’t change this property, it follows that \( \vec{T}(\pi) \times \vec{T}'(\pi) \) is also a normal vector to the osculating plane. Ignoring scalars once again, we obtain a normal vector:
\[ (1, -1, -\pi) \times (\pi, 3\pi + \pi^3, -\pi^2 - 4) = ((\pi^2 + 2)^2, 2(\pi^2 + 2), \pi(\pi^2 + 2)) = (\pi^2 + 2)(\pi^2 + 2, 2, \pi). \]

Ignoring scalars, we can use \( (\pi^2 + 2, 2, \pi) \) as a normal vector, and we get an equation of the osculating plane:
\[ (\pi^2 + 2)(x - \pi) + 2(y - \pi) + \pi z = 0. \]
4. (a) (i) This limit does not exist:

\[
\lim_{(x,x) \to (0,0)} \frac{x^3 \cos x}{x^2 + x^4} = \lim_{x \to 0} \frac{x}{1 + x^2} = 0
\]

\[
\lim_{(y^2,y) \to (0,0)} \frac{y^4 \cos y}{y^4 + y^4} = \frac{1}{2}
\]

Since the limits above don’t coincide, the overall limit does not exist.

(ii) First, note that we have

\[
\lim_{z \to 0} \frac{\cos z - 1}{z^2} = \lim_{z \to 0} \frac{-\sin z}{2z} = \lim_{z \to 0} \frac{-\cos z}{2} = -\frac{1}{2}.
\]

Moreover,

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2 + 1}{x^2 - y^2 + 2} = \frac{1}{2}.
\]

Therefore, the limit exists, and \(\lim_{(x,y,z) \to (0,0,0)} \frac{(x^2 + y^2 + 1)(\cos z - 1)}{(x^2 - y^2 + 2)(z^2)} = -\frac{1}{4}\).

(iii) If \(p = 1\), we need to show that \(\lim_{(x,y) \to (0,0)} \frac{xy}{x^4 + y^4}\) does not exist. That is because \(\lim_{(x,x) \to (0,0)} \frac{x^2 + y^2}{x^4 + y^4} = \lim_{x \to 0} \frac{1}{x^2} = +\infty\), but \(\lim_{(x,0) \to (0,0)} \frac{0}{x^2} = 0\).

If \(p = 2\), we need to show that \(\lim_{(x,y) \to (0,0)} \frac{x^2y^2}{x^4 + y^4}\) does not exist. That is because \(\lim_{(x,x) \to (0,0)} \frac{x^4 + y^4}{x^4 + y^4} = \frac{1}{2}\), but \(\lim_{(x,0) \to (0,0)} \frac{0}{x^2} = 0\).

If \(p > 3\), then \((xy)^p\) is has degree \(2p > 4\), so the limit exists and \(\lim_{(x,y) \to (0,0)} \frac{(xy)^p}{x^4 + y^4} = 0\).

(b) A level surface has the form \(f(x, y, z) = k\), so

\[x + y^2 + z^2 = k.\]

For any \(k\), as above, the level surface is an elliptic paraboloid.

(c)

\[g(x, y, z) = (x + y^2 + z^2)e^{y^2 + z}\]

\[\frac{\partial g}{\partial x} = 1 \cdot e^{y^2 + z} + (x + y^2 + z^2) \cdot 0 = e^{y^2 + z}\]

\[\frac{\partial g}{\partial y} = (2y) \cdot e^{y^2 + z} + (x + y^2 + z^2) \cdot e^{y^2 + z} \cdot (2y) = 2y(x + y^2 + z^2 + 1)e^{y^2 + z}\]

\[\frac{\partial g}{\partial z} = (2z) \cdot e^{y^2 + z} + (x + y^2 + z^2)e^{y^2 + z} \cdot 1 = e^{y^2 + z}(x + y^2 + z^2 + 2z)\]

\[\frac{\partial^2 g}{\partial x \partial z} = e^{y^2 + z} \cdot 1 = e^{y^2 + z}\]
5. (a) 

\[ f(x, y, z) = x^2 - 2xy + 7z^2 - 15 \]

\[ \nabla f(x, y, z) = (2x - 2y, -2x, 14z) \]

\[ \nabla f(4, 1, -1) = \langle 8 - 2, -8, -14 \rangle = \langle 6, -8, -14 \rangle \]

This means that the vector \( \langle 6, -8, -14 \rangle \) is the normal to the tangent plane, so an equation for the tangent plane is:

\[ 6(x - 4) - 8(y - 1) - 14(z + 1) = 0 \]

\[ 3x - 4y - 7z = 15 \]

(b) The line perpendicular to the tangent plane has direction \( \langle 6, -8, -14 \rangle \), so a parametric equation for it is:

\[ (4, 1, -1) + t(6, -8, -14) = (6t + 4, -8t + 1, -14t - 1) \]

This gives the corresponding symmetric equations:

\[ \frac{x - 4}{6} = \frac{y - 1}{-8} = \frac{z + 1}{-14}. \]

(c) For this part, we need to estimate the possible error in the volume measurement:

\[ V = 4(x - \frac{1}{2})yz + \frac{2}{3}y^2z^2 \]

\[ dV = 4(x - \frac{1}{2})ydz + 4(x - \frac{1}{2})zdy + 4yzdx + \frac{4}{3}y^2dz \]

Since we have a possible error of .01 meter in each direction, the possible error for the volume is:

\[ dV = 4 \cdot 3.5 \cdot 10 \cdot 0.01 + 4 \cdot 3.5 \cdot 3 \cdot 0.01 + 4 \cdot 10 \cdot 3 \cdot 0.01 + \frac{4}{3} \cdot 10 \cdot 3^2 \cdot 0.01 + \frac{4}{3} \cdot 10^2 \cdot 3 \cdot 0.01 \]

\[ dV = 8.82 \]

When \((x, y, z) = (4, 10, 3)\), we get \( V = 1020 \). Even with a maximum error of 8.22, the volume would still be greater than 600, which means the submarine can accommodate all 6 people.