PRACTICE MIDTERM 1 SOLUTIONS

FALL 2018

UN1201: CALCULUS III, SECTIONS 6 & 7

Problem 1.

(a) False: consider the counterexample with $\vec{u} = \langle 1, 0, 0 \rangle, \vec{v} = \langle 1, 1, 0 \rangle, \vec{w} = \langle 1, 1, 1 \rangle$; in this case,

$$(\vec{u} \times \vec{v}) \times \vec{w} = \langle 0, 0, 1 \rangle \times \vec{w} = \langle -1, 1, 0 \rangle$$
$$\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times \langle 1, -1, 0 \rangle = \langle 0, 0, -1 \rangle,$$

so the vector cross product is not associative!

(b) True: any plane P containing the line ℓ that P, Q, and R lie on will contain the three points and in particular, any rotation of P around ℓ will produce a new plane containing P, Q, and R.

(c) True: calculating the dot products $\vec{QP} \cdot \vec{QR}$, $\vec{RQ} \cdot \vec{RP}$, and $\vec{PR} \cdot \vec{PQ}$ together with dot product / cosine formula, we can find that

$$\angle PQR = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 62^\circ < 90^\circ$$
$$\angle QRP = \cos^{-1}\left(\frac{4}{\sqrt{7}}\right) \approx 55^\circ < 90^\circ$$
$$\angle RPQ = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 62^\circ < 90^\circ.$$

(d) False: observe that the second inequality is equivalent to $z > 2\sqrt{x^2 + y^2}$ which is the same as the first, but z can be negative (without one of the negative signs, the assertion would be true because we would have that $z = \pm 2\sqrt{x^2 + y^2 + 3}$ whose absolute value is always greater than $2\sqrt{x^2 + y^2}$.

Problem 2.

Date: October 2, 2018.

(a) Observe that

$$\vec{PQ} \times \vec{QR} = \langle 3, 8, 0 \rangle \times \langle -12, -2, -5 \rangle = \langle -40, 15, 90 \rangle \neq \langle 0, 0, 0 \rangle .$$

Then $\vec{PQ} \times \vec{QR}$ cannot be parallel by the cross product / sine formula, i.e. P, Q, and R are not collinear.

Since $\mathcal{P}$ contains P, Q, and R, it also contains $\overline{PQ}$ and $\overline{QR}$. In particular, $\vec{PQ} \times \vec{QR}$ is perpendicular to these lines and is a normal vector for $\mathcal{P}$. Using the point P and $\vec{PQ} \times \vec{QR}$, we can write $\mathcal{P}$ in point-normal form as

$$0 = -40(x-5) + 15(y+4) + 90(z-7) = -40x + 15y + 90z - 370$$

Dividing both sides by 5, we obtain

$$0 = -8x + 3y + 18z - 74.$$

(b) We have $\vec{PQ} = \langle 3, 8, 0 \rangle$ and $\vec{u} = \langle -1, -1, 2 \rangle$. Then the projection is

$$\text{proj}_{\vec{u}}\left(\vec{PQ}\right) = \frac{\vec{PQ} \cdot \vec{u}}{||\vec{u}||^2}\vec{u} = \frac{-11}{6}\langle -1, -1, 2 \rangle = \left\langle \frac{11}{6}, \frac{11}{6}, -\frac{11}{3} \right\rangle$$

(c) In (a), we had that $\vec{PQ} \times \vec{QR} = \langle -40, 15, 90 \rangle$ was normal to $\mathcal{P}$. To produce a unit normal vector, we can take

$$\vec{n} = \frac{\langle -40, 15, 90 \rangle}{||\langle -40, 15, 90 \rangle||} = \frac{\langle -40, 15, 90 \rangle}{5\sqrt{397}} = \left\langle -\frac{8}{\sqrt{397}}, \frac{3}{\sqrt{397}}, \frac{18}{\sqrt{397}} \right\rangle.$$

Then the other unit normal vector to $\mathcal{P}$ is

$$-\vec{n} = -\left\langle -\frac{8}{\sqrt{397}}, \frac{3}{\sqrt{397}}, \frac{18}{\sqrt{397}} \right\rangle = \left\langle \frac{8}{\sqrt{397}}, -\frac{3}{\sqrt{397}}, -\frac{18}{\sqrt{397}} \right\rangle.$$

The line through R parallel to $\vec{n}$ is given by

$$\begin{aligned} R + t\vec{n} &= \langle -4, 2, 2 \rangle + t\left\langle -\frac{8}{\sqrt{397}}, \frac{3}{\sqrt{397}}, \frac{18}{\sqrt{397}} \right\rangle \\ &= \left\langle -4 - t\frac{8}{\sqrt{397}}, 2 + t\frac{3}{\sqrt{397}}, 2 + t\frac{18}{\sqrt{397}} \right\rangle. \end{aligned}$$

Problem 3.

(a) Recall that $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$ in cylindrical coordinates. Then we have

$$2r\cos(\theta) - r\sin(\theta) + 3z = 10.$$

(b) We can multiply both sides by ρ and use $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ to get

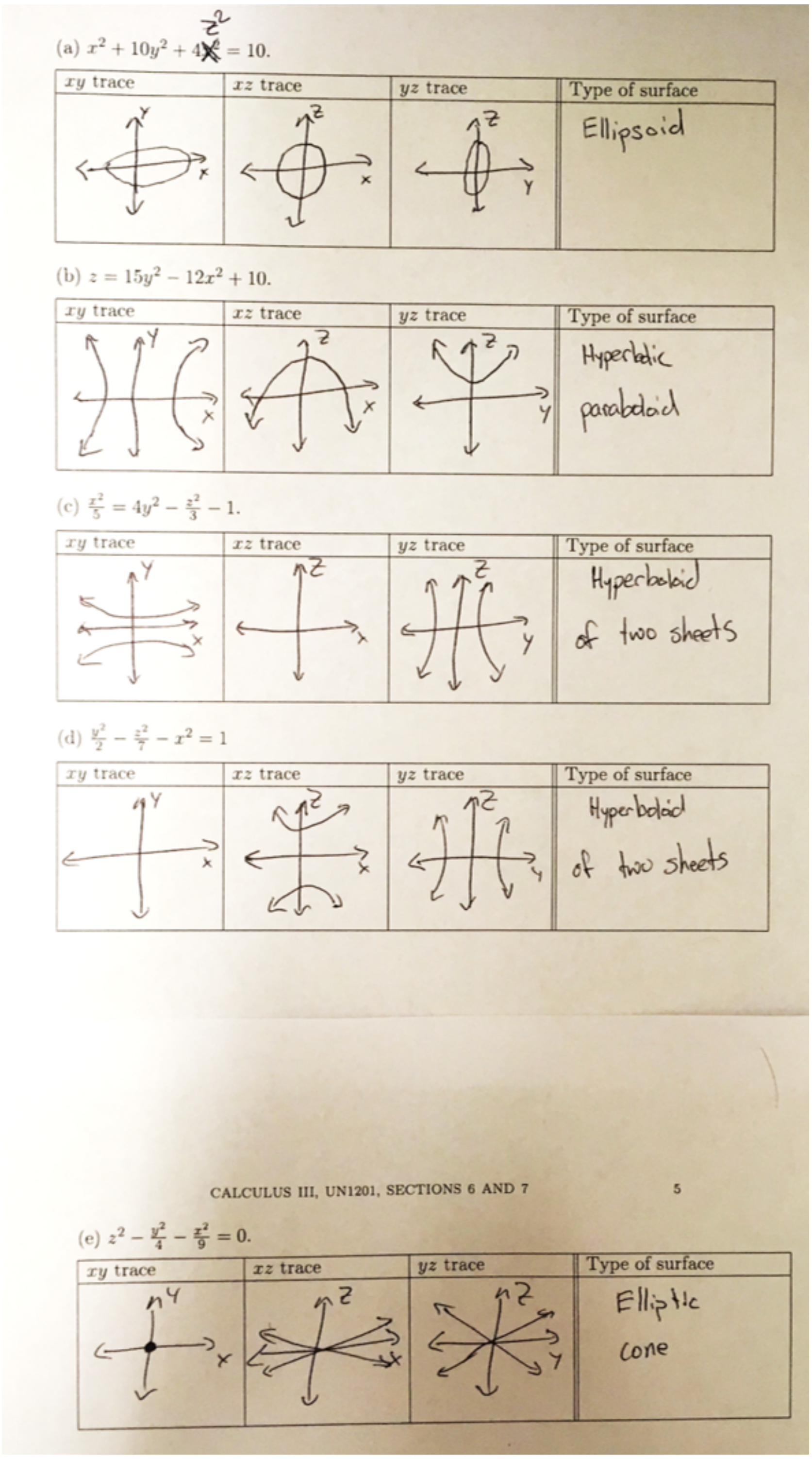
$$\begin{aligned}\rho^2\cos^2(\varphi) &= \rho^2\cos(2\theta)\sin^2(\varphi)\\ &= \rho^2(\cos^2(\theta) - \sin^2(\theta))\sin^2(\varphi)\\ &= \rho^2\cos^2(\theta)\sin^2(\varphi) - \rho^2\sin^2(\theta)\sin^2(\varphi).\end{aligned}$$

Then if we convert from spherical coordinates to Cartesian coordinates via $x = \rho\cos(\theta)\sin(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, and $z = \rho\cos(\varphi)$,

$$z^2 = x^2 - y^2.$$

Rearranging as $x^2 = y^2 + z^2$, we see that this is a cone!

Problem 4.



(a) $x^2 + 10y^2 + 4\overset{z^2}{\cancel{x^2}} = 10.$

xy trace	xz trace	yz trace	Type of surface
			Ellipsoid

(b) $z = 15y^2 - 12x^2 + 10.$

xy trace	xz trace	yz trace	Type of surface
			Hyperbolic paraboloid

(c) $\frac{x^2}{5} = 4y^2 - \frac{z^2}{3} - 1.$

xy trace	xz trace	yz trace	Type of surface
			Hyperboloid of two sheets

(d) $\frac{y^2}{2} - \frac{z^2}{7} - x^2 = 1$

xy trace	xz trace	yz trace	Type of surface
			Hyperboloid of two sheets

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(e) $z^2 - \frac{y^2}{4} - \frac{x^2}{9} = 0.$

xy trace	xz trace	yz trace	Type of surface
			Elliptic cone

Problem 5.

(a) The line ℓ will be parallel to the cross product of the normal vectors of $\mathcal{P}_1$ and $\mathcal{P}_2$, which is $\langle 1,-1,2\rangle \times \langle 2,1,-1\rangle = \langle -1,5,3\rangle$. Observe that the point $(0,8,6)$ lies on both $\mathcal{P}_1$ and $\mathcal{P}_2$. Then ℓ can be given by the parametric equation $(0,8,6)+t(-1,5,3) = (-t, 8+5t, 6+3t)$, i.e. $x=-t, y=8+5t, z=6+3t$.

(b) Since $\mathcal{P}_3$ is parallel to $\mathcal{P}_1$, it will have the same normal vector and therefore have the form $x-y+2z=k$ for some k. Since Q lies on $\mathcal{P}_3$, we have that $k=(0)-(1)+2(0)=-1$. Thus, $\mathcal{P}_3$ is given by $x-y+2z=-1$. We can see that R lies on $\mathcal{P}_3$ since $(3)-(0)+2(-2)=-1$.

(c) The line m is given by $(3t,0,-2t)$. Then the intersection of m and $\mathcal{P}_1$ can be found by solving

$$\begin{aligned}&(3t)-(0)+2(-2t)=4\\&-t=4\\&t=-4.\end{aligned}$$

Then the point on m given by $t=-4$ is $(-12,0,8)$. Similarly for $\mathcal{P}_3$,

$$\begin{aligned}&2(3t)+(0)-(-2t)=2\\&8t=2\\&t=\frac{1}{4}.\end{aligned}$$

Then the point on m given by $t=\frac{1}{4}$ is $(\frac{3}{4},0,-\frac{1}{2})$.

(d) Write $A=(-t_1, 8+5t_1, 6+3t_1)$ and $B=(3t_2,0,-2t_2)$. If A and B are the two points closest to each other, then $\vec{AB}$ must be perpendicular to both $\langle -1,5,3\rangle$ and $\langle 3,0,-2\rangle$ (the vectors parallel to the lines ℓ and m respectively). So in particular, $\vec{AB}\cdot\langle -1,5,3\rangle = 0$ and $\vec{AB}\cdot\langle 3,0,-2\rangle = 0$. So

$$\begin{aligned}0&=\langle 3t_2+t_1, -8-5t_1, -2t_2-3t_1\rangle\cdot\langle -1,5,3\rangle\\&=-35t_1-9t_2-40\\0&=\langle 3t_2+t_1, -8-5t_1, -2t_2-3t_1\rangle\cdot\langle 3,0,-2\rangle\\&=9t_1+13t_2.\end{aligned}$$

We can solve this system of equations by substituting $t_1 = -\frac{13}{9}t_2$ from the second equation into the first equation to get $t_2 = \frac{180}{187}$. Then $t_1 = -\frac{260}{187}$. So

$$A = \left(\frac{260}{187}, \frac{196}{187}, \frac{342}{187}\right)$$
$$B = \left(\frac{540}{187}, 0, -\frac{360}{187}\right).$$

The distance from A to B is $\left|\left|\overline{AB}\right|\right| = 2\sqrt{\frac{815}{187}} \approx 4.175$.

We find $\vec{AB} = \left\langle \frac{280}{187}, -\frac{196}{187}, -\frac{702}{187} \right\rangle$. Using A as our base point, we get the following symmetric equations for the line passing through A and B:

$$\frac{x - \frac{260}{187}}{\frac{280}{187}} = \frac{y - \frac{196}{187}}{-\frac{196}{187}} = \frac{z - \frac{342}{187}}{-\frac{702}{187}}.$$