CALCULUS III, UN1201, SECTIONS 6 AND 7

Practice Final

Please write your name on *each* page.

No electronic devices (laptops, calculators, telephones) or notes are allowed during exams. Numerical expressions involving square roots, trigonometric functions, or inverse trigonometric functions need not be simplified and should not be converted into decimals.

Show all work. To receive full credit, you must justify your answers.

Use the back side of the page if you need more space to do a problem.

1	2	3	4	5

Total:

Some possibly useful formulas

The curvature of the curve given by the vector function $\vec{r}$ is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$

If $\vec{r}$ is given by $\vec{r}(t) = \langle t, f(t), 0 \rangle$ for some function f, so the curve it parametrizes is the graph of the function $y = f(x)$ in the plane, then the curvature at the point $(x, f(x), 0)$ is

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

1. True or false? 5 points for each correct answer **with correct justification**, 0 points for wrong answer, 2 points for "don't know."

(a) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three vectors. Suppose $\vec{u} \cdot \vec{v} = 1$ and $\vec{u} \cdot \vec{w} = 2$. Then $\vec{u}$ is perpendicular to $2\vec{v} - \vec{w}$. (Do not use coordinates for these vectors! The answer must be based on the algebra of vectors alone.)

(b) Let C be the graph of the function $y = f(x)$ in the plane, and suppose the curvature of C at every point equals 0. Then C is a straight line.

(c) Let S be an ellipsoid in 3-space and let $f(x, y, z)$ be a differentiable function in three variables. Suppose P is the point on S where f takes its maximum value. Then the gradient of f is parallel to the tangent plane to S at P.

(d) The equalities $0 \leq x^2y^2 \leq x^4 + y^4$ are true for all (x, y).

2. Let ℓ be the line with symmetric equations

$$x-2=\frac{3-y}{2}=\frac{1-z}{3}.$$

Let m be the line with symmetric equations

$$x-3=\frac{y+4}{3}=\frac{z-2}{c}$$

where c is a number to be determined in the next part of the problem.

(a) Find c so that the lines m and ℓ intersect. Show that there is only one possibility for c.

(b) With c as in part (a), find the plane that contains the two lines.

3. Let $f(x, y) = x^3 - xy + y^2 - x$. Let R be the set of (x, y) with $x \geq 0$, $y \geq 0$, $x + y \leq 2$.

(a) (3 points) Sketch the region R and verify that it's the interior of a triangle.

(b) (5 points) Find the critical points of f and determine which ones belong to R.

(c) (12 points) Find the absolute minimum and absolute maximum of f on R.

4. (a) Identify the type of the level surfaces $f(x, y, z) = k$ of the function $f(x, y, z) = x^2 - 2y^2 + \frac{z^2}{9} + 14$, as k varies.

(b) Find the equation of the tangent plane to the level surface $f(x, y, z) = 16$ at the point $(1, 2, 9)$.

5. (a) Let f and be the functions defined for (x, y, z) in 3-space, $(x, y, z) \neq (0, 0, 0)$, by the following formulas:

(i) $f(x, y, z) = \frac{x^2 - y^2 + z^2}{x^2 + y^2 + z^2}$; (ii) $g(x, y, z) = \frac{x^4 + y^4 - z^3}{x^2 + y^2 + z^2}$.

One of the limits below exists; which one is it?

$$\lim_{(x,y,z)\to(0,0,0)} f(x, y, z); \quad \lim_{(x,y,z)\to(0,0,0)} g(x, y, z).$$

Justify your answer.

(b) The weight (in tons) of a certain truck is given by the formula

$$w(x, y, z) = 3\sqrt{1 + \frac{(xy)^2}{200} + \frac{xyz}{6}}$$

where x is the length, y is the width, and z is the height. The dimensions (in meters) have been measured as follows: $x_0 = 15$; $y_0 = 4$, $z_0 = 3$. But the measurements are only accurate to .14 meters. The truck has to drive across a bridge whose maximum load is known to be 22.5 tons. Can we be sure the bridge will not collapse? Explain your answer.

6. Let $\vec{r}(t) = \langle 2t, t^2, ln(t)\rangle$ for $t > 0$.

(a) Show that the curve C parametrized by $\vec{r}(t)$ passes through the points $P = (2, 1, 0)$ and $Q = (4, 4, ln(2))$ and find its arc length between these points.

(b) Compute the curvature of C as a function of t.

(c) Find the equation of the osculating planes to C at the points P and Q and determine equations for their line of intersection.

7. (a) Let $f(x, y, z) = x^2 - xy + z^3$. Use the chain rule to find the partial derivatives $\frac{\partial f}{\partial \rho}$, $\frac{\partial f}{\partial \theta}$, and $\frac{\partial f}{\partial \phi}$ with respect to the spherical coordinates ρ, θ, ϕ.

(b) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ on the surface $f(x, y, z) = 3$ at the point $(2, 1, 1)$.

(c) Find the maximum rate of change of the function f at the point $(2, 1, 1)$ and the direction in which it occurs.

8. (a) Find the roots of the quadratic equation

$$x^2 - 4x + 10 = 0.$$

Is either of the roots a real number?

(b) Compute

(i) $\frac{3-2i}{4+3i}$; (ii) $-2+2i^{\frac{1}{3}}$.

(c) Use the complex exponential to find a formula for $\cos(3\theta)$.

9. (a) Find all the critical points of the function $f(x, y) = \sin(xy)$.

(b) For each critical point found in (a), determine whether it's a local maximum, a local minimum, or a saddle point.

10. Find the points (x, y, z) at which the function

$$f(x, y, z) = x^2 + 2xz + y^2/4 + z^2 - 3$$

takes its minimum on the elliptic paraboloid

$$z = x^2 + 3y^2.$$