ARITHMETIC OF THE OSCILLATOR REPRESENTATION

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§1. Moduli

1.1. Let $V = \mathbb{Q}^{2g}$, and let $<,>$ be the standard symplectic form on V, represented by the matrix $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. Let T be the lattice $\mathbb{Z}^{2g} \subset V$. We extend $<,>$ to the finite adeles $V \otimes \mathbf{A}_f$. Let $G = Sp(V,<,>)$. For any additive character

$$\psi : \mathbf{A}_f \to \mu_\infty \subset \mathbb{G}_m,$$

the pairing

$$(x,y) \mapsto < x,y >_\psi \overset{def}{=} \psi(< x,y >)$$

is a bilinear form on $V \otimes \mathbf{A}_f$, with values in μ_∞. Moreover, $T \otimes \hat{\mathbb{Z}}$ is a maximal subgroup $U \subset V \otimes \mathbf{A}_f$ with the property that

$$< t_1, t_2 >_\psi = 1 \ \forall t_1, t_2 \in U.$$

If $\psi : \mathbf{A}_f \to \mu_\infty$ is another character, then $\psi(z) = \psi(az)$, for some $a \in \mathbf{A}^{f,\times}$. On the other hand, the action of $Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ on $\mu_\infty \subset \mathbb{G}_m$ defines an action on $Hom(\mathbf{A}_f, \mathbb{G}_m)$. There exists $\sigma \in Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ such that $\psi' = \psi^\sigma$. If we denote the reciprocity isomorphism

$$\hat{\mathbb{Z}} \xrightarrow{\sim} Gal(\mathbb{Q}^{ab}/\mathbb{Q})$$

$$a \mapsto \sigma_a$$

normalized so that p goes to the inverse of Frobenius (mod p), then evidently

$$\psi(az) = \psi^{\sigma_a^{-1}}(z) \quad \forall \psi \in Hom(\mathbf{A}_f, \mathbb{G}_m). \tag{1.1.1}$$

1.2. Let $N \geq 3$ be a positive integer, and let $\mathcal{M}_N^g$ be the moduli space of g-dimensional principally polarized abelian varieties A with level N structure

$$\alpha_N : A[N] \xrightarrow{\sim} N^{-1}T/T \subset V \otimes \mathbf{A}_f/T.$$

We require that the restriction to $N^{-1}T/T$ of the bilinear form $<,>_\psi$ on $V \otimes \mathbf{A}_f/T$ correspond via α, up to a scalar multiple in $(\mathbb{Z}/N\mathbb{Z})^\times$, to the Weil pairing on $A[N]$ induced by the given polarization. Choose $\psi \in Hom(\mathbf{A}_f/\hat{\mathbb{Z}}, \mathbb{G}_m)$, and let $\mathcal{M}_{N,\psi}^g \subset \mathcal{M}_N^g$ be the connected component for which α is a symplectic isomorphism, with respect to $<,>_\psi$ on the right hand side.

Since $N \geq 3$, there exists a universal abelian scheme $\mathcal{A}_N^g/\mathcal{M}_N^g$, with principal polarization $[\Lambda]^g$, and a symplectic similitude of finite group schemes

$$\alpha_N^g : \mathcal{A}_N^g[N] \xrightarrow{\sim} (N^{-1}T/T)_{\mathcal{M}_N^g}.$$

1.3. Let $\pi : \mathcal{A}^g_N \to \mathcal{M}^g_N$ be the canonical map. With ψ as in 1.2, let $\mathcal{A}^g_{N,\psi} = \pi^{-1}(\mathcal{M}^g_{N,\psi})$. We let

$$\mathcal{M}^g = \varprojlim_N \mathcal{M}^g_N, \ \mathcal{M}^g_\psi = \varprojlim_N \mathcal{M}^g_{N,\psi},$$

define $\mathcal{A}_g$ to be the pullback to $\mathcal{M}^g$ of $\mathcal{A}^g_N/\mathcal{M}^g_N$, for any N, and define $\mathcal{A}^g_\psi$ likewise.

1.4. Let S be any scheme over $Spec(\mathbb{Q})$, and let A be an abelian scheme of relative dimension g over S. The family $\mathcal{T} = \mathcal{T}(A)$ of homomorphisms $B/S \to A/S$, with kernel finite and flat over S, is an inverse system for which

$$[n] = (multiplicationbyn) : A \to A$$

forms a cofinal subset. The inverse limit over $\mathcal{T}$ exists in the category of schemes; we denote it $\hat{A}$. Lacking a better name, we call $\hat{A}$ the *isocompletion* of A. We let

$$\hat{A}(t) = \varprojlim_{B \in \mathcal{T}(A)} B(tors).$$

A full level structure on A (or on $\hat{A}$) is an isomorphism over S

$$\alpha : \hat{A}(t) \xrightarrow{\sim} V \otimes \mathbf{A}_{f\,S} \tag{1.4.1}$$

Obviously any finite flat isogeny $A/S \to A'/S$ induces an isomorphism $\hat{A} \xrightarrow{\sim} \hat{A}'$. Following Deligne, we define the category of *abelian varieties up to isogeny*, or a.v.i., to be the category whose objects are abelian schemes over S and in which isogenies are isomorphisms. The functor $A \mapsto \hat{A}$ takes the category of abelian varieties over $\mathbb{Q}$ to the category of schemes over $\mathbb{Q}$; the essential image of this functor is equivalent to the category of a.v.i.

More generally, if $d \in \mathbb{Z}$, we can consider the family $\mathcal{T}_d$ (resp. $\mathcal{T}^d$ of $B/S \to A/S \in \mathcal{T}$ whose kernel is of order equal to a product of primes dividing d (resp of order prime-to-d); we can define $\hat{(A)}_{(d)}$ (resp. $\hat{A}^{(d)}$ to be the inverse limit over $'CalT_d$ (resp. $\mathcal{T}^d$). If we define the category of abelian schemes over S up to d-primary isogeny (resp. prime-to-d isogeny) to be the category of abelian varieties with isogenies of degree equal to a product of primes dividing d (resp. prime to d) then $A \mapsto \hat{A}_{(d)}$ and $A \mapsto \hat{A}^{(d)}$ take the category of abelian schemes over $\mathbb{Z}[d^{-1}]$ (resp. over the localization of $\mathbb{Z}$ at d) to the respective categories of abelian schemes up to isogeny.

For any abelian variety A, let $NS(A)$ be the Neron-Severi group of A. Then $NS(A) \otimes \mathbb{Q}$ depends only on the isogeny class of A. A polarized a.v.i. is an a.v.i. A together with an element of $(NS(A) \otimes \mathbb{Q})/\mathbb{Q}^*$.

1.5. We apply these remarks to the situation considered in 1.2, 1.3. There is a universal full level structure on $\hat{\mathcal{A}}^g$:

$$\alpha^g : hat\mathcal{A}^g(t) \xrightarrow{\sim} V \otimes \mathbf{A}_{f\,\mathcal{M}^g} \tag{1.5.1.}$$

Let $\psi \in Hom(\mathbf{A}_f, \mathbb{G}_m)$, and let $\hat{\mathcal{A}}^g_\psi$ be the restriction of $\hat{\mathcal{A}}^g$ to $\mathcal{M}^g_\psi$. Now, $\mathcal{M}^g_\psi$ is the moduli space for polarized a.v.i. with full level structures (1.5.1) of type ψ, in the obvious sense, and the morphism $\hat{\mathcal{A}}^g_\psi \to \mathcal{M}^g_\psi$ may be viewed as the universal polarized a.v.i. with full level structure of type ψ.

1.6. Let $GSp = GSp(V, <, >)$ be the group of symplectic similitudes of V. The group $GSp(\mathbf{A}_f)$ acts naturally on the moduli space $\mathcal{M}^g_\psi$ through its action on full level structures. In fact, a geometric point of $\mathcal{M}^g_\psi$ is given by a triple (A, Λ, α), where A is an a.v.i., Λ is a polarization on A, and α is a full level structure (1.4.1). Thus if $\gamma \in GSp(\mathbf{A}_f)$ and $x = (A, \Lambda, \alpha) \in \mathcal{M}^g_\psi$, then $\gamma(x) = (A, \Lambda, \gamma \circ \alpha)$.

Similarly, a geometric point of $\hat{\mathcal{A}}^g_\psi$ is given by a quadruple (A, a, Λ, α), where $a \in \hat{A}$. Thus the action of $GSp(\mathbf{A}_f)$ on $\mathcal{M}^g_\psi$ lifts to an action on $\hat{\mathcal{A}}^g_\psi : \gamma(A, a, \Lambda, \alpha) = (A, a, \Lambda, \gamma \circ \alpha)$, for all $\gamma \in GSp(\mathbf{A}_f)$.

2. Linear systems

Most of the contents of §2 and §3 are copied verbatim from Mumford's series of articles "On the equations defining abelian varieties." Not all of our assertions are literally stated as such by Mumford, but the reader will easily supply the missing details.

The following definitions and results are mostly in §1 and §6 of [M1]. Let S be a scheme over $Spec\mathbb{Q}$. The basic object in this section is a pair (A, L), where $\pi : A \to S$ is an abelian scheme of relative dimension g, and L is a relatively ample invertible sheaf (= line bundle) on A. One should think of S as the moduli space of polarized abelian varieties with level structure, although eventually more general cases will be considered.

Let T be a scheme over S, and let $H(L)(T)$ be the group of sections $\alpha : T \to A_T$ such that, if $T_\alpha : A_T \to A_T$ is translation by α, then

$$T^*_\alpha L \cong L \otimes \pi^*_T M$$

for some invertible sheaf M on T; here $\pi_T : A_T \to T$ is the natural morphism. Let $H_0(L)(T)$ be the subgroup of $\alpha \in H(L)$ such that

$$T^*_\alpha L \cong L.$$

Finally, let $\mathcal{G}(L)(T)$ be the group of pairs (α, φ), where $\alpha \in H_0(L)(T)$ and $\varphi : T^*_\alpha L \xrightarrow{\sim} L$ is an isomorphism. Then $T \mapsto H(L)(T)$ and $T \mapsto \mathcal{G}(L)(T)$ are functors from the category of S-schemes to the category of groups. Mumford shows in [M1], II, p. 76, that these functors are representable by group schemes $\underline{\mathcal{G}}(L)$ and $\underline{H}(L)$, respectively, flat and of finite presentation over S, which fit into an exact sequence in the Zariski topology:

$$1 \to (\mathbb{G}_m)_S \to \underline{\mathcal{G}}(L) \to \underline{H}(L) \to 1. \tag{2.1}$$

Moreover, $\underline{H}(L)$ is a finite flat subgroup scheme of A.

The group scheme $\underline{\mathcal{G}}(L)$ is a version of the Heisenberg group. It acts naturally on the sheaf $\pi * L$: If $U \subset S$ is an open subset, $\gamma = (\alpha, \varphi) \in \underline{\mathcal{G}}(L)(U)$, and $\varphi \in \Gamma(A_U, L)$, then

$$\varphi^\gamma \stackrel{def}{=} \varphi \circ T^*_\alpha(f) \in \Gamma(A_U, L) \tag{2.2}$$

This action can be understood most easily when S is the spectrum of a local ring R. Then $\underline{H}(L)$ is the constant group scheme $H(L) \times S$. In general, the extension

(2.1) defines a skew symmetric pairing on $\underline{H}(L)$ with values in $\mathbb{G}_m$: if $\tilde{x}, \tilde{y} \in \underline{\mathcal{G}}(L)$ map to $x, y \in \underline{H}(L)$, then

$$e^L(x,y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} \in \mathbb{G}_m \tag{2.3}$$

is well-defined and non-degenerate. When $S = SpecR$ as above, then we may decompose $\underline{H}(L)$ as a product

$$\underline{H}(L) \cong H^0 \times \hat{H}^0, \tag{2.4}$$

where H^0 is a maximal isotropic subgroup for e^L, and $hatH^0 = Hom(H^0, \mathbb{G}_m)$.

On the other hand, we may define the Heisenberg group $\mathcal{G}(H^0)$ to be the set of triples

$$\{(t, h, \hat{h}) \mid \ t \in \mathbb{G}_m, h \in H^0, \hat{h} \in \hat{H}^0$$

with multiplication

$$(t, h, \hat{h}) \cdot (t`, h`, \hat{h}`) = (t \cdot t` \cdot \hat{h}'(h), h + h`, \hat{h} + \hat{h}'). \tag{2.5}$$

An isomorphism as in (2.4) induces an isomorphism

$$\mathcal{G}(L) \overset{def}{=} \underline{\mathcal{G}}(L)(S) \cong \mathcal{G}(H^0). \tag{2.6}$$

It follows from a version of the Stone-Von Neumann theorem, and from results of Mumford in §1 of [M1], that

Proposition 2.7 (Mumford). *The R-module $\Gamma(A,L)$ is free of rank $|H^0| = \sqrt{|H(L)|}$. The representation (2.1.2) of $\mathcal{G}(L)$ on $\Gamma(A,L)$ is identified, via (2.6), with the unique irreducible representation of $\mathcal{G}(H^0)$ on which its center $\mathbb{G}_m$ acts by the identity character*

$$t \mapsto \textit{(multiplication by } t\textit{)}.$$

Although H^0 is not uniquely determined by L, its set of elementary divisors is:

$$H^0 \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z}.$$

for some g-tuple of integers $\delta = (d_1, \ldots, d_g)$, with $d_1 \mid d_2 \mid \cdots \mid d_g$. We say L is of *type* δ; the type of a relatively ample line bundle is constant on an irreducible scheme S over $Spec(\mathbb{Z})$ provided d_g is invertible on S. We let

$$d = d(\delta) = \prod_i d_i.$$

Then if (A, L) and $\pi : A \to S$ are as above, the sheaf π_*L is locally free of rank d on S.

Let

$$K(\delta) = \bigoplus_{i=1}^{g} \mathbb{Z}/d_i\mathbb{Z},$$

with δ as above; let $\hat{K}(\delta) = Hom(K(\delta), \mathbb{G}_m)$, and let $H(\delta) = K(\delta) \times \hat{K}(\delta)$. We let $\mathcal{G}(\delta) = \mathcal{G}(K(\delta))$, with multiplication as in (2.5). Following Mumford, we define a θ-structure on (A, L) to be an isomorphism

$$\underline{\mathcal{G}}(L) \xrightarrow{\sim} \mathcal{G}(\delta)_S$$

which restricts to the identity on their respective centers $\mathbb{G}_m$. Of course, the algebraic group $\mathcal{G}(\delta)$ has a representation ρ_δ on the free $\mathbb{Z}[d^{-1}]$-module

$$V_\delta = V_\delta(\mathbb{Z}[d^{-1}]) = \{\text{functions from } K_\delta \text{ to } \mathbb{Z}[d^{-1}]\} :$$

If $f \in V_\delta$, $\alpha \in \mathbb{G}_m(\mathbb{Q})$, $x \in K(\delta)$, $\ell \in \hat{K}(\delta)$, then

$$\rho_\delta((\alpha, x, \ell))f(y) = \alpha \cdot \ell(y) \cdot f(x+y). \tag{2.8}$$

We paraphrase a proposition of Mumford ([M1], II, p. 80):

Proposition 2.9 (Mumford). *Let (A, L) be of type δ, and let*

$$\lambda : \underline{\mathcal{G}}(L) \xrightarrow{\sim} \mathcal{G}(\delta)_S$$

be a θ-structure on (A, L). Denote by ρ^λ the representation of $\mathcal{G}(\delta)_S$ on π_L defined by (2.2) and the isomorphism λ. Let $\mathbb{K} = \mathbb{K}(L, \lambda) \subset \pi_L$ be the subsheaf on which $\rho^\lambda(\hat{K}(\delta))$ acts trivially. Define an action of $\mathcal{G}(\delta)_S$ on $V_\delta \otimes \mathbb{K}$ by tensoring ρ^λ with the trivial action on $\mathbb{K}$. Then there is an isomorphism of sheaves

$$\pi_L \xrightarrow{\sim} V_\delta \otimes \mathbb{K} \tag{2.10}$$

equivariant with respect to $\mathcal{G}(\delta)_S$, and unique up to multiplication by an element of $\Gamma(S, \mathcal{O}_S^*)$.

2.11. Suppose now that $(\sigma : B \to S, M)$ is another polarized abelian scheme over S, and suppose we are given an isogeny $p : A \to B$ over S, with kernel K flat over S, and an isomorphism $L \xrightarrow{\sim} p^*M$. In [M1], p. 290 ff., Mumford constructs a subgroup scheme $\tilde{K} = \tilde{K}(L, M) \subset \underline{\mathcal{G}}(L)$ which fits into a diagram

$$\begin{array}{ccccc}
\underline{\mathcal{G}}(L) & \longrightarrow & \underline{H}(L) & \longrightarrow & 0 \\
\cup & & \cup & & \\
\tilde{K} & \xrightarrow{\sim} & K & &
\end{array}$$

and a canonical isomorphism $Z(\tilde{K})/\tilde{K} \xrightarrow{\sim} \mathcal{G}(M)$, where $Z(\tilde{K})$ is the centralizer of $\tilde{K}$ in $\underline{\mathcal{G}}(L)$.

The isomorphism $K \xrightarrow{\sim} \tilde{K} \subset \underline{\mathcal{G}}(L)$ defines an action of K on L; this action is the descent datum corresponding to the isomorphism $L \xrightarrow{\sim} p * M$. In particular, the canonical morphism $\sigma_* M \to \pi_* L$ provides an isomorphism $\sigma_* M \xrightarrow{\sim} (\pi * L)^{\tilde{K}}$ (cf. [M1], §1, proof of Theorem 4). On the other hand, in characteristic prime to d, there is a natural projection

$$P(L, M) : (\pi_* L)^{\tilde{K}} \to (\pi * L)^{\tilde{K}} \cong \sigma_* M.$$

Let $R(L,M) = KerP(L,M) \subset \pi_*L$; $R(L,M)$ is a locally free subsheaf of π_*L.

Suppose (B,M) is of type η, and let

$$\begin{array}{ccc} \lambda : \underline{\mathcal{G}}(L) & \xrightarrow{\sim} & \mathcal{G}(\delta)_S \\ \cup & & \cup \\ Z(\tilde{K}) & \xrightarrow{\sim} & Z(\lambda(\tilde{K})) \\ \downarrow & & \downarrow \\ \mu : \underline{\mathcal{G}}(M) & \xrightarrow{\sim} & \mathcal{G}(\eta)_S \end{array} \tag{2.11.1}$$

be a commutative diagram where the horizontal arrows are θ-structures and the left vertical arrow is given by the isomorphism $Z(\tilde{K})/\tilde{K} \xrightarrow{\sim} \underline{\mathcal{G}}(M)$. We may naturally identify $V_\eta \cong V_\delta^{\lambda(\tilde{K})}$. Proposition 2.9 then gives us a commutative diagram of sheaves over S, equivariant under $Z(\lambda(\tilde{K}))$:

$$\begin{array}{ccc} \pi_*L & \xrightarrow{\sim} & V_\delta \otimes \mathbb{K}(L,\lambda) \\ \downarrow & & \downarrow \\ \sigma_*M & \xrightarrow{\sim} & V_\eta \otimes \mathbb{K}(M,\mu). \end{array} \tag{2.11.2}$$

Here the left vertical arrow is $P(L,M)$ and the left vertical arrow is of the form $P(\delta,\eta) \otimes t$, where $P(\delta,\eta) : V_\delta \to V_\delta^{\lambda(\tilde{K})} \cong V_\eta$ is the projection and $t : \mathbb{K}(L,\lambda) \to \mathbb{K}(M,\mu)$ makes the diagram commute. Note that the horizontal arrows are determined only up to multiplication by an element of $\Gamma(S, \mathcal{O}_S^*)$.

Now L and M are both relatively ample, hence define morphisms $A \to \mathbb{P}_S(\pi_*L)$, $B \to \mathbb{P}_S(\sigma_*M)$. It follows immediately from the ampleness of M that the image of A in $\mathbb{P}_S(\pi_*L)$ does not intersect $\mathbb{P}_S(R(L,M))$. The homomorphism $P(L,M)$ determines a morphism $\mathbb{P}_S(\pi_*L) - \mathbb{P}_S(R(L,M)) \to \mathbb{P}_S(\sigma_*M)$. Let $R(\delta,\eta) = Ker\ P(\delta,\eta)$. We obtain a commutative diagram of schemes over S:

$$\begin{array}{ccccc} A & \xrightarrow{\sim} & \mathbb{P}_S(\pi_*L) - \mathbb{P}_S(R(L,M)) & \xrightarrow{\sim} & (\mathbb{P}(V_\delta) - \mathbb{P}(R(\delta,\eta)) \times S \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & \mathbb{P}_S(\sigma_*M) & \xrightarrow{\sim} & \mathbb{P}(V_\eta) \times S. \end{array} \tag{2.11.3}$$

3. Polarized towers

We want to work out the adelic analogue of the constructions in §2. Although Mumford only concerns himself with the 2-adic theory, there is no difficulty in applying his methods to the general p-adic case. Indeed, at no point do we make use of the deeper parts of Mumford's theory.

We begin by rigidifying the problem, although this is not essential at this point. Let $\pi : A \to S$ be as in §2.

Definition 3.1. *A (relative) line bundle L over A is* **symmetric** *if there is an isomorphism*

$$\iota : L \xrightarrow{\sim} (-1)^* L$$

where (-1) denotes multiplication by (-1) on A. The isomorphism ι induces an isomorphism

$$\iota_x : L_x \xrightarrow{\sim} L_x$$

for each point x of order 2 in A. We say L is totally symmetric if ι can be chosen so that ι_x is the identity for all $x \in A[2]$.

3.2. Remarks. (a) Here we have identified $A[2]$ with the set of geometric points of the finite flat group scheme $[Ker(2) : A \to A]$ over S, where (2) denotes multiplication by 2.

(b) If L is any line bundle, then $L \otimes (-1)^*(L)$ is totally symmetric.

(c) The most important property of totally symmetric line bundles is the following: If L and L' are algebraically equivalent totally symmetric line bundles on A, then there exists a line bundle M on S such that

$$L \cong L' \otimes \pi^* M.$$

(d) On p. 78 of [M1, II], Mumford defines a canonical involution

$$\delta_{-1} : \underline{\mathcal{G}}(L) \to \underline{\mathcal{G}}(L)$$

for any symmetric line bundle L. On the other hand, there is an obvious involution $D_{-1} : \mathcal{G}(\delta) \to \mathcal{G}(\delta)$, for any δ:

$$D_{-1}(\alpha, x, \ell) = (\alpha, -x, -\ell), \alpha \in \mathbb{G}_m,\ x \in K(\delta),\ \ell \in \tilde{K}(\delta).$$

A θ-structure $\lambda : \underline{\mathcal{G}}(L) \xrightarrow{\sim} \mathcal{G}(\delta)_S$ is symmetric if

$$\lambda \circ \delta_{-1} = D_{-1} \circ \lambda.$$

3.3. A *tower* (resp. a *prime-to-N tower*, for some positive integer N) is an inverse system of abelian schemes over S

$$\mathcal{T} = \{\pi_\alpha : A_\alpha \to S\}_{\alpha \in \Sigma}$$

indexed by a partially ordered set Σ, and isogenies $p_{\alpha\beta} : A_\alpha \to A_\beta$ (resp. isogenies of degree prime to N) whenever $\alpha > \beta$, satisfying the compatibility condition $p_{\beta\gamma} \circ p_{\alpha\beta} = p_{\alpha\gamma}$. We also require that $\mathcal{T}$ satisfy the following saturation condition: if $A_\alpha \in \mathcal{T}$ and B is an abelian scheme over S admitting an isogeny (resp. prime-to-N isogeny) $p : B \to A_\alpha$ (resp. $p : A_\alpha \to B$) then B is isomorphic to some A_β with $\beta > \alpha$ (resp. $\alpha > \beta$) in such a way that p corresponds to $p_{\beta\alpha}$ (resp. $p_{\alpha\beta}$). To avoid trivialities, we assume deg $p_{\alpha\beta} > 1$ whenever $\alpha > \beta$. We also do not require that the $\{A_\alpha\}$ be distinct; indeed, we require that the isogeny (n) = multiplication by n:

$$(n) : A_\alpha \to A_\alpha$$

be an isogeny in the tower (resp. provided $(n, N) = 1$). Finally, we assume that, if $\alpha > \beta_1, \beta_2$, and $K_i = Ker(p_{\alpha\beta_i})$, $i = 1, 2$, then $K_1 \subset K_2 \Leftrightarrow \beta_1 \geq \beta_2$. We let $\hat{\mathcal{T}} = \hat{A}_\alpha$, for any $\alpha \in \Sigma$, where $\hat{A}_\alpha$ is the isocompletion defined in 1.4.

A *polarized tower* $(\mathcal{T},\mathcal{P})$ is a tower $\mathcal{T}=\{A_\alpha\}$ and a collection L_α/A_α of symmetric relatively ample line bundles, $\forall\alpha\in\Sigma$, with isomorphisms

$$\varphi_{\alpha\beta}: p*_{\alpha\beta}(L_\beta)\xrightarrow{\sim}L_\alpha$$

satisfying $\varphi_{\alpha\beta}\circ\varphi_{\beta\gamma}=\varphi_{\alpha\gamma}$. We require furthermore that L_α be totally symmetric for all but finitely many α. We also make the following saturation requirement: Let $A_\alpha\in\mathcal{T}$; let B be an abelian scheme over S admitting an isogeny $p:B\to A_\alpha$ (resp. $p:A\alpha\to B$); one can also place restrictions on the degree as in the previous paragraph. Suppose there exists a line bundle L_B/B such that $p*L_\alpha\simeq L_B$ (resp. $p*L_B\simeq L_\alpha$). As above, we have an isomorphism $\sigma:B\xrightarrow{\sim}A_\beta$, for some $\beta\in\Sigma$, and we assume that $L_B\simeq\sigma*(L_\beta)$. Given an isogeny $p\alpha\beta$, we let $\tilde{K}_{\alpha\beta}=\tilde{K}(L_\alpha,L_\beta)$ be the subgroup scheme of $\underline{\mathcal{G}}(L_\alpha)$ defined in 2.11; then $Z(\tilde{K}_{\alpha\beta})/\tilde{K}_{\alpha\beta}\simeq\underline{\mathcal{G}}(L_\beta)$.

The symbol $\mathcal{P}$ above is used to denote the polarization, or the set $\{L_\alpha\}_{\alpha\in\Sigma}$. The hypothesis of saturation implies easily (cf. [M1], I, p. 293) that L_α is of degree 1 for (exactly) one $\alpha\in\Sigma$.

Any pair (A,L) over S, with L totally symmetric and relatively ample, generates a polarized tower over S in the obvious way. If (A',L') is isogenous to (A,L) , in the sense that there exists an isogeny $p:A\to A'$ such that $p*(L')\simeq L$, then the towers generated by (A,L) and (A',L') are isomorphic. Thus we may speak of an *isogeny class* of polarized towers.

3.4. If $(\mathcal{T},\mathcal{P})$ is a polarized tower, then the locally free sheaves $\pi_{\alpha,*}(L_\alpha)$ form an inverse system over S:

$$\varphi_{\alpha\beta,*}:\pi_{\alpha,*}(L_\alpha)\to(\pi_{\alpha,*}(L_\alpha))^{\tilde{K}_{\alpha\beta}}\xrightarrow{\sim}\pi_{\beta,*}(L_\beta)$$

and we may take the inverse limit

$$\pi_*(\mathcal{P})=\varprojlim_\alpha\pi_{\alpha,*}(L_\alpha).$$

The fiber at $x\in S$ of $\pi_*(\mathcal{P})$ is just the inverse limit of the spaces of sections of all the line bundles L_α/A_α.

Let $V^f(\mathcal{T})=\hat{A}_\alpha(t)$, for any $\alpha\in\Sigma$, (notation (1.4.1)), viewed as a sheaf in the Zariski topology on S. Then $V^f(\mathcal{T})$ is canonically isomorphic to the étale homology group

$$V^t(\mathcal{T})\simeq(\pi_\ell R^{-1}\pi_{\alpha,*}(\mathbb{Z}_\ell))\otimes\mathbb{Q}$$

for *any* $\alpha\in\Sigma$. Assuming $\mathcal{T}$ has a polarization $\mathcal{P}$ as above, represent $x\in V^f(\mathcal{T})$ as a limit

$$x\;=\;(x_\alpha)_{\alpha\in\Sigma}\quad x_\alpha\in A_\alpha(tors),$$

and let

$$\Sigma^x=\{\alpha\in\Sigma\mid\; x_\alpha\in\underline{H}(L_\alpha)\}.$$

We define the Heisenberg group scheme of $(\mathcal{T},\mathcal{P})$ to be the group scheme $\mathcal{G}(\mathcal{P})$ whose T-valued points, for T a scheme over S, are pairs

$$(x,\{\varphi_\alpha\}_{\alpha\in\Sigma_x})\quad x=(x_\alpha)\in V^f(\mathcal{T}),$$

$$\varphi_\alpha : T^*_{x_\alpha}(L_\alpha) \xrightarrow{\sim} L_\alpha.$$

We require the isomorphisms φ_α to satisfy natural compatibility conditions, as in [M1], §7. There is an exact sequence of sheaves in the Zariski topology

$$1 \to \mathbb{G}_{m,S} \to \mathcal{G}(\mathcal{P}) \to V^f(\mathcal{T}) \to 1. \tag{3.5}$$

For any $\alpha \in \Sigma$, the lattice $K(\alpha) = \ker(\hat{A}_\alpha(t) \to A_\alpha(tors)) \subset \hat{A}_\alpha(t) \simeq V^f(\mathcal{T})$ lifts canonically to a subgroup scheme $\tilde{K}(\alpha) \subset \mathcal{G}(\mathcal{P})$. In [M1], II, p. 103 ff. (where our $\tilde{K}(\alpha)$ is denoted $K(\alpha)$), Mumford shows that $Z(\tilde{K}(\alpha))/\tilde{K}(\alpha)$, where $Z(\tilde{K}(\alpha))$ is the centralizer of $\tilde{K}(\alpha)$ in $\mathcal{G}(\mathcal{P})$, is canonically isomorphic to $\underline{\mathcal{G}}(L_\alpha)$.

In order to continue, we provisionally choose an isomorphism

$$\psi : \mathbf{A}_f/\hat{\mathbb{Z}} \xrightarrow{\sim} \varinjlim_n \mu_n \overset{\text{def.}}{=} \mu_\infty.$$

Let $\mathcal{G}(g,\psi)$ be the standard Heisenberg group on the finite adeles:

$$\mathcal{G}(g,\psi) = \{(\alpha, x, \ell) \mid \ \alpha \in \mathbb{G}_m, x, \ell \in (\mathbf{A}_f)^g\},$$

with multiplication

$$(\alpha_1, x_1, \ell_1) \cdot (\alpha_2, x_2, \ell_2) \ = \ (\alpha_1 \cdot \alpha_2 \cdot \psi({}^t x_1 \cdot \ell_2), x_1 + x_2, \ell_1 + \ell_2).$$

In analogy with (2.8), there is a natural representation

$$\rho_\psi : \mathcal{G}(g,\psi) \to Aut(\mathcal{S}^g)$$

where$\mathcal{S}^g$ is the *Schwartz space* of locally constant, compactly supported $\mathbb{Q}^{ab}$-valued functions on $(\mathbf{A}_f)^g$. (The groups $Aut(\mathcal{S}^g)$ and $\mathcal{G}(g,\psi)$, as well as the representation ρ_ψ, have interpretations in the category of schemes over $\mathbb{Q}$: cf. Appendix to §4.)

3.7. The comparison of ρ_ψ with the actions on finite levels is worked out in detail in [M1], II, pp. 110-111. Let $U \subset V$ be two lattices in $\mathbb{Q}^g$; let $\hat{U} = U \otimes \mathbf{A}_f \subset (\mathbf{A}_f)^g$, $\hat{V} = V \otimes \mathbf{A}_f \subset (\mathbf{A}_f)^g$. Define $\mathcal{S}^g(U,V)$ to be the space of functions on $(\mathbf{A}_f)^g$ with support in $\hat{V}$, constant modulo $\hat{U}$. We say $(U',V') \subset (U,V)$ if $U \subset U' \subset V' \subset V$; then we have maps

$$Res : \mathcal{S}^g(U,V) \to \mathcal{S}^g(U,V');\ Tr : \mathcal{S}^g(U,V') \to \mathcal{S}^g(U',V'),$$

where Res is restriction of functions on V to functions on V', and $Tr(f) = \int_{U'/U} f(u)du$, where du is Haar measure on U'/U with $\int_{U'/U} du = 1$. In terms of these two maps, we may define

$$\hat{\mathcal{S}}^g = \varprojlim_{U \subset V} \mathcal{S}^g(U,V).$$

Given (U,V) as above, we may identify $\hat{V}/\hat{U} \simeq K(\delta)$ for some δ as in §2; in terms of this identification, $\mathcal{S}^g(U,V)$ is isomorphic to V_δ. Let $K(U,V) = \{1\} \times \hat{U} \times \times \hat{V}^\perp \subset \mathcal{G}(g,\psi)$, where

$$\hat{V}^\perp \ = \ \{y \in (\mathbf{A}_f)^g \mid \psi({}^t x \cdot y) = 1 \forall x \in \hat{V}\}.$$

Then $Z(K(U,V))/K(U,V) \simeq \mathcal{G}(\delta)$. One sees easily that $\mathcal{S}^g(U,V) = (\mathcal{S}^g)^{K(U,V)}$, and that the corresponding action of $\mathcal{G}(\delta) \simeq Z(K(U,V))/K(U,V)$ on $V_\delta \simeq \mathcal{S}^g(U,V)$ is just (2.8).

Suppose $g(U',V') \subset (U,V)$ and $\hat{V}'/\hat{U}' \simeq K(\mu)$. The map $Tr{\circ}Res : \mathcal{S}^g(U,V) \to \mathcal{S}^g(U',V')$ coincides with projection onto the $K(U',V')$-invariant subspace. We have an action of

$$\varprojlim_{U\subset V} Z(K(U,V))/K(U,V) \simeq \mathcal{G}(g,\psi) \text{ (cf. [M1], II, p. 103)}$$

on $\hat{\mathcal{S}}^g = \varprojlim \mathcal{S}^g(U,V)$ induced from the actions of $\mathcal{G}(\delta) \simeq Z(K(U,V))/K(U,V)$ on $V_\delta \simeq \mathcal{S}^g(U,V)$ for all (U,V); this action is exactly ρ_ψ.

As in the case of finite level, we have the following data:

(3.8) A skew symmetric pairing $e^{\mathcal{P}} : V^f(\mathcal{T}) \otimes V^f(\mathcal{T}) \to \mathbb{G}_m$;

(3.9) involutions $\delta_{-1} : \mathcal{G}(\mathcal{P}) \xrightarrow{\sim} \mathcal{G}(\mathcal{P})$, $D_{-1} : \mathcal{G}(g,\psi) \xrightarrow{\sim} \mathcal{G}(g,\psi)$.

3.10. A symmetric θ-structure on $(\mathcal{T},\mathcal{P})$, of type ψ, is an isomorphism

$$c : \mathcal{G}(\mathcal{P}) \xrightarrow{\sim} \mathcal{G}(g,\psi)$$

which restricts to the identity on the mutual center $\mathbb{G}_m$ and which satisfies

$$c \circ \delta_{-1} = D_{-1} \circ c.$$

As explained in [M1],II, p. 106, there is a one-to-one correspondence between symmetric θ-structures of type ψ, and full level structures

$$\beta : V^f(\mathcal{P}) \xrightarrow{\sim} (\mathbf{A}_f)^g \times (\mathbf{A}_f)^g,$$

under which the bilinear pairing $e^{\mathcal{P}}$ on the left-hand side corresponds to the pairing

$$(x_1,y_1) \otimes (x_2,y_2) \mapsto \psi({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1).$$

Such a level structure β will be called *symplectic* (of type ψ).

In analogy with Proposition 2.9, we have

3.11. Proposition. *: Let $(\mathcal{T},\mathcal{P})$ be a polarized tower, and let*

$$c : \mathcal{G}(\mathcal{P}) \xrightarrow{\sim} \mathcal{G}(g,\psi)$$

be a symmetric θ-structure of type ψ. There is a unique line bundle $\mathbb{K}/S$, and an isomorphism of $\mathcal{G}(g,\psi)$-modules

$$\pi_*(\mathcal{P}) \xrightarrow{\sim} \hat{\mathcal{S}}^g \otimes_{\mathbb{Q}^{ab}} \mathbb{K}$$

where $\mathcal{G}(g,\psi)$ acts trivially on $\mathbb{K}$ and through ρ_ψ (resp., through the isomorphism c^{-1}) on $\mathcal{S}^g$ (resp., on $\pi_*(\mathcal{P})$. This isomorphism is unique up to multiplication by a scalar in $\Gamma(S,\mathcal{O}_S^*)$.

Proof. : One simply takes the limit over the corresponding θ-structures at finite level, checking that they are compatible by using 3.7 and the commutativity of (2.11.2). We observe that the existence of a symmetric θ-structure of type ψ over S implies that S is a scheme over $Spec(\mathbb{Q}^{ab})$. We will account for this below.

4. Construction of the bundle of forms of weight $\frac{1}{2}$

Let $N \geq 3$ be a positive integer, and define $\mathcal{M}_N^g$, $\pi : \mathcal{A}_N^g \to \mathcal{M}_N^g$, $[\Lambda]^g$, and α_N^g as in §1. The polarization $[\Lambda]^g$ defines at each fiber $\mathcal{A}_{N,x}^g$ of $\mathcal{A}_N^g/\mathcal{M}_N^g$ an algebraic equivalence class $[\Lambda]_x^g$ of ample line bundles on $\mathcal{A}_{N,x}^g$ of degree one. For any $\Lambda \in [\Lambda]_x^g$, let $\mathcal{L}_{N,x}^g = \Lambda \otimes (-1)^*\Lambda$; then $\mathcal{L}_{N,x}^g$ depends only on the algebraic equivalence class $[\Lambda]_x^g$, and is totally symmetric (cf. 3.2). This construction globalizes to define a relatively ample, totally symmetric line bundle $\mathcal{L}_N^g$ over $\mathcal{A}_N^g$. A priori, $\mathcal{L}_N^g$ is determined only up to tensoring with the pullback of a line bundle on $\mathcal{M}_N^g$. Let $\varepsilon : \mathcal{M}_N^g \to \mathcal{A}_N^g$ be the zero section. We normalize $\mathcal{L}_N^g$ by requiring that

$$\varepsilon^*(\mathcal{L}_N^g) \simeq \mathcal{O}_{\mathcal{M}_N^g};$$

then $\mathcal{L}_N^g$ is determined uniquely up to isomorphism.

Pick a positive integer $d \geq 3$, and let $L = L^{g,d}$ be the pullback of $\mathcal{L}_N^g$ under multiplication by d. Let δ be the type of L. We assume N large enough so that

$$\underline{H}(L^{\times 2}) \subset \mathcal{A}_N^g[N], \tag{4.1}$$

and we assume d is the largest integer for which (4.1) holds.

Using ψ, we may define an isomorphism, for any $d_0 \in \mathbb{Z}$:

$$F_\psi : \mathbb{Z}/d_0\mathbb{Z} \xrightarrow{\sim} \hat{\mathbb{Z}}/d_0\mathbb{Z};\ F_\psi(a)(b) = \psi(d_0^{-1}ab),\ a, b \in \mathbb{Z}/d_0\mathbb{Z}.$$

In this way, we may define likewise

$$F_\psi : K(\delta) \xrightarrow{\sim} \hat{K}(\delta).$$

We then have a group $\mathcal{G}(\delta, \psi)$, whose points are given by $\mathbb{G}_m \times K(\delta) \times K(\delta)$, with multiplication induced from 2.5 via F_ψ.

The remarks in [M1] I, pp. 317-320 imply that the hypothesis (4.1) allows us to define a unique symmetric θ-structure

$$\beta : \underline{\mathcal{G}}(L) \xrightarrow{\sim} \mathcal{G}(\delta, \psi)_{\mathcal{M}_{N,\psi}^g}$$

which reduces mod centers to the restriction of α_N^g to $\underline{H}(L) \subset \underline{H}(L^2)$. It thus follows from Proposition 2.9 that there exists a canonically defined line bundle $\Theta_{N,\psi}$ over $\mathcal{M}_{N,\psi}^g$, and an isomorphism

$$Sch : \pi_*L \otimes \Theta_{N,\psi} \xrightarrow{\sim} V_\delta \otimes \mathcal{O}_{CalM_{N,\psi}^g}. \tag{4.2}$$

We recall that Sch intertwines the action of $\underline{\mathcal{G}}(L)$ on π_*L with the action of $\mathcal{G}(\delta, \psi)$ on V_δ, and is uniquely determined up to a scalar in $\Gamma(\mathcal{M}_{N,\psi}^g, \mathcal{O}_{CalM_{N,\psi}^g})$, which is isomorphic to $\mathbb{G}_m$ except when $g = 1$ (the case $g = 1$ will be taken care of ??? when we discuss Fourier expansions, in §7 below).

We let $L_\Theta = L \otimes \pi^*\Theta_{N,\psi}$. Then L_Θ is relatively very ample, since $d \geq 3$, and we have an imbedding over $\mathcal{M}_{N,\psi}^g$:

$$\mathcal{A}_{N,\psi}^g \hookrightarrow \mathbb{P}_{\mathcal{M}_{N,\psi}^g}(\pi_*L \otimes \Theta_{N,\psi}) \xrightarrow{\sim} \mathbb{P}(V_\delta) \times \mathcal{M}_{N,\psi}^g, \tag{4.3}$$

where the second isomorphism is Sch.

Denote by $j$the composite of (4.3) with projection on the first factor:

$$j : \mathcal{A}^g_{N,\psi} \to \mathbb{P}(V_\delta).$$

Then

$$(j \circ \varepsilon) * \mathcal{O}(1)_{\mathbb{P}(V_\delta)} \simeq \Theta_{N,\psi}. \tag{4.4}$$

The content of [M1], §6, is the identification of the image of $j \circ \varepsilon$ in $\mathbb{P}(V_\delta)$. For us, the important point is a much more elementary fact: namely, that the image of $j \circ \varepsilon$ is a moduli space of a slightly lower level (on the order of $N/2$). This is not really difficult to make explicit, but is suffices for our purposes to work in the limit.

Thus, let $\mathcal{M}^g$, $\mathcal{A}^g$ be as in 1.3, and define $\mathcal{L}^g$ likewise. Define $L^{g,d}$ as above. The polarized abelian scheme $(\mathcal{A}^g, \mathcal{L}^g)$ defines a polarized tower, in the sense of 1.2, in which the pairs $(^g, L^{g,d})$, $d = 1, 2, \ldots$ form a set of cofinal objects. We denote this polarized tower $(\mathcal{T}^g, \mathcal{P}^g)$ and define $\hat{\mathcal{A}}^g$ as in 1.4.

Let $\hat{\mathcal{A}}^g_\psi$ be the restriction of $\hat{\mathcal{A}}^g$ to $\mathcal{M}^g_\psi$. There is a universal full level structure on $\hat{\mathcal{A}}^g_\psi$:

$$\alpha_g : \hat{\mathcal{A}}^g_\psi(tors) \xrightarrow{\sim} V \otimes \mathbf{A}_{f\,\mathcal{M}^g_\psi} \tag{4.5}$$

under which the skew symmetric pairings $e^{\mathcal{P}}$ and $<,>_\psi$ correspond. Let

$$\pi_*\mathcal{P} = \varprojlim_d \pi_* L^{g,d}$$

as above. As in 3.10, we have a unique symmetric θ-structure $\beta : \mathcal{G}(\mathcal{P}) \xrightarrow{\sim} \mathcal{G}(g, \psi)$ compatible with α_g. Proposition 3.11 provides us with a line bundle Θ_ψ on $\mathcal{M}^g_\psi$, and an equivariant isomorphism

$$Sch : \pi_*\mathcal{P} \otimes \Theta_\psi \xrightarrow{\sim} \hat{\mathcal{S}}^g \otimes \mathcal{O}_{\mathcal{M}^g_\psi}.$$

This induces an imbedding

$$\hat{\mathcal{A}}^g_\psi \hookrightarrow \mathbb{P}_{\mathcal{M}^g_N,\psi}(\pi_*\Theta_\psi) \xrightarrow{\sim} \mathbb{P}(\hat{\mathcal{S}}^g) \times \mathcal{M}^g_\psi \tag{4.6}$$

Let $\vartheta_\psi : \hat{\mathcal{A}}^g_\psi \to \mathbb{P}(\hat{\mathcal{S}}^g)$ denote projection on the first factor. The composite of the identity section $\varepsilon : \mathcal{M}^g_\psi \to \hat{\mathcal{A}}^g_\psi$, followed by ϑ_ψ, defines a morphism also denoted $\vartheta_\psi : \mathcal{M}^g_\psi \to \mathbb{P}(\hat{\mathcal{S}}^g)$. The following theorem follows directly from the remarks on p. 82 of [M1], II:

Theorem 4.7 (Mumford). *: The morphism* $\vartheta_\psi : \mathcal{M}^g_\psi \to \mathbb{P}(\hat{\mathcal{S}}^g$ *is a locally closed immersion; the bundle* $\Theta_\psi = \vartheta^*_\psi(\mathcal{O}_{\mathbb{P}(\hat{\mathcal{S}}^g)}(1))$.

4.8. The action of $\mathcal{G}(g,\psi)$ on $\hat{\mathcal{S}}^g$ induces a projective representation $\mathbb{P}(\omega_\psi)$ of $G(\mathbf{A}_f)$ on $\hat{\mathcal{S}}^g$, first defined by Weil. For our purposes it is most convenient to work at finite levels. Let

$$B_0(g,\psi) = \{\gamma \in Aut(\mathcal{G}(g,\psi)) \mid \ \gamma \text{ restricts to the identity on } \mathbb{G}_m\}.$$

Now $B_0(g,\psi)$ is canonically the semi-direct product of $G(\mathbf{A}_f)$ and the group of inner automorphisms of $\mathcal{G}(g,\psi)$. (Actually in [W], pp. 180-183, 188, Weil shows that $B_0(g,\psi)$ is the semi-direct product of $\mathcal{G}(g,\psi)$ and the *pseudosymplectic group* over $\mathbf{A}_f$, but since we are working in characteristic zero, the pseudosymplectic group and $G(\mathbf{A}_f)$ are canonically isomorphic. His calculations make sense in the category of algebraic groups over $\mathbb{Q}$, as described in the appendix to §4.)

Thus $G(\mathbf{A}_f)$ acts naturally on $\mathcal{G}(g,\psi)$, and for $\gamma \in G(\mathbf{A}_f)$, we can define a representation

$$\rho_\psi^\gamma : \mathcal{G}(g,\psi) \to Aut(\hat{\mathcal{S}}^g) : \rho_\psi^\gamma(\sigma) = \rho_\psi(\gamma^{-1}\sigma).$$

This can be interpreted in terms of finite levels as follows. Let $U \subset (\mathbf{A}_f)^g$ be a lattice, and let $\Gamma(U) \subset G(\mathbf{A}_f)$ be the stabilizer of the lattice $U \times U \subset (\mathbf{A}_f)^g \times (\mathbf{A}_f)^g$. Let $\mathcal{S}(U)$ be the Schwartz space of U, and let $\mathcal{G}(U,\psi) = \mathbb{G}_m \times U \times U^\perp \subset \mathcal{G}(g,\psi)$, with multiplication law (3.6). We denote by $\rho_{\psi,U}$ the natural representation of $\mathcal{G}(U,\psi) on \mathcal{S}(U)$.

Now the map $G(\mathbf{A}_f) \to B_0(g,\psi)$ takes $\Gamma(U)$ to the stabilizer of $\mathcal{G}(U,\psi)$. As above, each $\gamma \in \Gamma(U)$ defines a representation $\rho_{\psi,U}^\gamma$ on $\mathcal{S}(U)$. By an analogue of the Stone-Von Neumann theorem (or a generalization of Proposition 2.7), one sees that $\rho_{\psi,U}$ and $\rho_{\psi,U}^\gamma$ are equivalent irreducible representations. The corresponding projective representations are thus canonically isomorphic. We thus obtain a unique morphism $\mathbb{P}(\omega_{\psi,U})(\gamma) : \mathbb{P}(\mathcal{S}(U)) \to \mathbb{P}(\mathcal{S}(U))$ which intertwines $\rho_{\psi,U}$ and $\rho_{\psi,U}^\gamma$; $\mathbb{P}(\omega_{\psi,U})$ is a projective representation of $\Gamma(U)$.

On the other hand, $\Gamma(U) = \Gamma(n^{-1}U)$, $n = 1,2,\ldots$. We thus obtain a compatible system of projective representations $\mathbb{P}(\omega_\psi, n^{-1}U)$ of $\Gamma(U)$, $n = 1,2,\ldots$, where the map $\mathcal{S}(n^{-1}U) \to \mathcal{S}(m^{-1}U)$, for $m \div n$, is given by restriction of functions. In the limit, we have a projective representation $\mathbb{P}(\omega_\psi,(U))$ of $\Gamma(U)$ on $\varinjlim_n (n^{-1}U) \simeq \hat{\mathcal{S}}^g$; for any $\gamma \in \Gamma(U)$, $\mathbb{P}(\omega_\psi,(U))$ intertwines the restrictions to $\mathcal{G}(U,\psi)$ of $\rho_p si$ and ρ_ψ^γ.

Now every element of $G(\mathbf{A}_f)$ belongs to $\Gamma(U)$ for some lattice U. If $\gamma \in \Gamma(U) \cap \Gamma(V)$, then the projective representations $\mathbb{P}(\omega_{\psi,(U)})$ and $\mathbb{P}(\omega_{\psi,(V)})$ both intertwine $\rho_\psi \mid_{\mathcal{G}(U\cap V)}$ and $\rho_\psi^\gamma \mid_{\mathcal{G}(U\cap V)}$, hence coincide. In this way we obtain a projective representation $\mathbb{P}(\omega_\psi)$ of $G(\mathbf{A}_f)$ on $\hat{\mathcal{S}}^g$.

In [W], Weil proves that $\mathbb{P}(\omega_\psi)$ (or rather the analogous representation on the Schwartz space) lifts to a genuine representation of a double cover of $G(\mathbf{A}_f)$. In general, we prefer to work with the tautological representation associated with the projective representation:

$$\omega_\psi : \tilde{G}(\mathbf{A}_f) \to Aut(\hat{\mathcal{S}}^g).$$

Here $\tilde{G}(\mathbf{A}_f)$ is an extension of $G(\mathbf{A}_f)$ by $\mathbb{G}_m$ defined by the projective representation $\mathbb{P}(\omega_\psi)$. Thus $\tilde{G}(\mathbf{A}_f)$ acts on $\mathcal{O}_{\mathbb{P}(\hat{\mathcal{S}}^g)}(1)$, extending the tautological action of $\mathbb{G}_m$, given by $t \mapsto$ multiplication by t. We explain in the appendix to §4, below, how $\tilde{G}(\mathbf{A}_f)$ may be regarded as an extension in the category of group schemes, and how to define the structure of algebraic variety on $\mathbb{P}(\hat{\mathcal{S}}^g)$.

Proposition 4.9. . *The action of $G(\mathbf{A}_f)$ on $\mathbb{P}(\hat{\mathcal{S}}^g)$ restricts to the canonical action on $\hat{\mathcal{A}}^g_\psi$. In particular, there is an action of $\tilde{G}(\mathbf{A}_f)$ on Θ_ψ which covers the action of $G(\mathbf{A}_f)$ on $\mathcal{M}^g_\psi$. The restriction of this action to the subgroup $\mathbb{G}_m$ is the tautological action.*

Proof. The proposition follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{G}(\mathcal{P}) & \longrightarrow & Aut(\pi_*\mathcal{P}) \\ {\scriptstyle \gamma\circ c}\downarrow & & \downarrow \\ \mathcal{G}(g,\psi) & \xrightarrow{\rho^\gamma_\psi\otimes 1} & Aut(\mathcal{S}^g\otimes_{\mathbb{Q}^{ab}}\mathbb{K}) \end{array} \tag{4.9.1}$$

for all $\gamma\in G(\mathbf{A}_f)$. Here c and the right-hand vertical arrow are as in Proposition 3.11, and $\gamma\circ c$ is c followed by the action of $\gamma\in G(\mathbf{A}_f)\subset B_0(g,\psi)$.

4.10. Remark. It is possible, a priori , that the extension $\tilde{G}(\mathbf{A}_f)$ depends on the choice of character ψ. There is an easy way to see that the isomorphism class of $\tilde{G}(\mathbf{A}_f)$ is independent of ψ. Let $\bar{B}_0(g,\psi)_1$ denote the normalizer in $Aut(\hat{\mathcal{S}}^g)$ of $\rho_\psi(\mathcal{G}(g,\psi))$. Let $\bar{B}_0(g,\psi)\subset\bar{B}_0(g,\psi)_1$ be the subgroup of elements α for which there exists an increasing filtration

$$\{0\}\subset U_1\subset U_2\subset\cdots\subset\bigcup_i U_i=(\mathbf{A}_f)^g$$

such that α stabilizes $\ker(\hat{\mathcal{S}}^g\to\mathcal{S}(U_i))$ for all i. There is an exact sequence of group schemes

$$1\to\mathbb{G}_m\to\bar{B}_0(g,\psi)\xrightarrow{\alpha}B_0(g,\psi)\to 1; \tag{4.10.1}$$

the surjectivity of α is equivalent to the existence of the projective representation $\mathbb{P}(\omega_\psi)$. We have $\tilde{G}(\mathbf{A}_f)=\alpha^{-1}(G(\mathbf{A}_f))\subset\bar{B}_0(g,\psi)$; it is canonically determined by $\mathcal{G}(g,\psi)$.

But the subgroup $\rho_\psi(\mathcal{G}(g,\psi))\subset Aut(\mathcal{S}^g)$ is independent of ψ. In fact, given a non-trivial ψ, every non-trivial additive character of $\mathbf{A}_f$ is of the form $\psi^t(x)=\psi(t^{-1}x)$, for some $t\in\mathbf{A}^{f,\times}$. Now the map $j^t:\mathcal{G}(g,\psi)\to\mathcal{G}(g,\psi^t)$, defined in the coordinates (3.7) by $j^t(\alpha,x,\ell)=(\alpha,x,t\ell)$, is an isomorphism of groups. Obviously

$$\rho_{\psi^t}(j^t(\alpha,x,\ell))=\rho_\psi((\alpha,x,\ell)),\forall(\alpha,x,\ell)\in\mathcal{G}(g,\psi). \tag{4.10.2}$$

It follows that $\bar{B}_0(g,\psi)$ and $\bar{B}_0(g,\psi^t)$ are canonically isomorphic, for every $t\in\mathbf{A}^{f,\times}$. Using the exact sequence (4.10.1), we see that $\tilde{G}(\mathbf{A}_f)$ is independent of the choice of ψ.

Appendix to §4: $\tilde{G}(\mathbf{A}_f)$ as an ind-group scheme over $\mathbb{Q}$

In order to explain how to interpret the exact sequence

$$1\to\mathbb{G}_m\to\tilde{G}(\mathbf{A}_f)\to G(\mathbf{A}_f)\to 1$$

as an extension of group schemes, we have first to describe the ind-group scheme structure on the totally disconnected group $G(\mathbf{A}_f)$. The procedure described here is valid for any locally compact totally disconnected group Γ, and provides such a Γ with the structure of group scheme over $Spec(\mathbb{Z})$.

Thus let Γ be any totally disconnected group, and let $\mathbb{Z}_\Gamma$ be the ring of locally constant compactly supported $\mathbb{Z}$-valued functions on Γ. Multiplication in the group Γ allows one to define a canonical coalgebra structure on $\mathbb{Z}_\Gamma$, making $Spec(\mathbb{Z}_\Gamma)$ into a group scheme.

Lemma 4.A.1. *The set Γ is canonically isomorphic to the set of global sections $Spec(\mathbb{Z}) \to Spec(\mathbb{Z}_\Gamma)$. The Zariski topology on the latter induces the usual topology of locally compact totally disconnected group on Γ.*

Proof. The lemma is obvious for Γ finite. Suppose $\Gamma = \varprojlim_i \Gamma_i$, with Γ_i finite for all i. If $C \subset \Gamma$ is a closed subset, then $C = \varprojlim_i C_i$, where $C_i \subset \Gamma_i$ is the set of zeroes of the ideal $\mathcal{J}_i \subset \mathbb{Z}_{\Gamma_i}$, say. Then C is obviously the set of zeroes of the ideal $\varinjlim_i \mathcal{J}_i \subset \mathbb{Z}_\Gamma$. Conversely, if $\mathcal{J} = \varinjlim_i \mathcal{J}_i$ is an ideal in $\mathbb{Z}_\Gamma$ with zero set C, then C is easily seen to be the inverse limit of the zero sets C_i of $\mathcal{J}_i$. The lemma follows for profinite Γ.

Now let $K \subset \Gamma$ be a profinite open subgroup, and let Σ be a set of coset representatives for Γ/K. Then we have

$$\Gamma = \coprod_{\sigma\in\Sigma} \sigma K;\ \mathbb{Z}_\Gamma = \sum_{\sigma\in\Sigma} \mathbb{Z}_{\sigma K} \tag{4.A.1.1}$$

where $\mathbb{Z}_{\sigma K} \simeq \mathbb{Z}_K$ is the ring of locally constant functions on σK. The lemma follows immediately from (4.A.1.1) and the special case of profinite groups.

4.A.2. In defining the projective space $\mathbb{P}(\hat{\mathcal{S}}^g)$ over $Spec(\mathbb{Q})$, we view the infinite dimensional vector space $\hat{\mathcal{S}}^g$ as the inverse limit of finite dimensional subspaces $\mathcal{S}^g(U, V)$, as in §3. Then the projective space $\mathbb{P}(\hat{\mathcal{S}}^g)$ over $Spec(\mathbb{Q})$ is the projective spectrum of the (non-noetherian) graded ring

$$\varinjlim_{U\subset V} \mathbb{Q}[\mathbb{Q}[V/U]] = \mathbb{Q}[\mathbb{Q}(\mathbf{A}_f)^g]].$$

4.A.3. We henceforward let $\Gamma = G(\mathbf{A}_f)$, and identify Γ with the group scheme $Spec(\mathbb{Q} \otimes \mathbb{Z}_\Gamma)$ over $\mathbb{Q}$. The projective representation of $G(\mathbf{A}_f)$ on $\hat{\mathcal{S}}^g$ is deduced from actions on finite levels, as in §4.10. It follows from Lemma 4.A.1 and the definition in 4.A.2 that

Lemma 4.A.4. *The $\mathbb{Q}^{ab}$-rational projective representation $\mathbb{P}(\omega_\psi)$ of $G(\mathbf{A}_f)$ on $\mathbb{P}(\hat{\mathcal{S}}^g)$ is a continuous action in the Zariski toplology.*

Standard considerations now imply

Proposition 4.A.5. *: There exists an ind-group scheme $\tilde{G}(\mathbf{A}_f)$ over $\mathbb{Q}$, an exact sequence*

$$1 \to \mathbb{G}_m \to \tilde{G}(\mathbf{A}_f) \to G(\mathbf{A}_f) \to 1$$

in the category of ind-group schemes , and a representation

$$\omega_\psi : \tilde{G}(\mathbf{A}_f) \to Aut(\hat{\mathcal{S}}^g),$$

continuous in the Zariski topology, such that $\omega_\psi(t) = t$, $t \in \mathbb{G}_m$*, and such that the induced action of* $G(\mathbf{A}_f)$ *on* $\mathbb{P}(\hat{\mathcal{S}}^g)$ *coincides with the projective representation* $\mathbb{P}(\omega_\psi)$ *defined in §4.*

5. Extending to GSp

Let $GSp = GSp(V, <,>)$ be the group of symplectic similitudes of V. We write $GSp = GSp^g$ when it is necessary to emphasize the dimension $2g$ of V. Let $\nu : GSp \to \mathbb{G}_m$ be the symplectic multiplier. Let $T \subset GSp$ be the subgroup

$$\{\gamma(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\} \subset GSp.$$

For $g \in G(\mathbf{A}_f)$, $\in \mathbf{A}^{f,\times}$, we let $g^t = \gamma(t)^{-1} \cdot g \cdot \gamma(t)$. We let ψ, ψ^t, and j^t be as in 4.10. Now $G(\mathbf{A}_f)$ acts on $\mathcal{G}(g, \psi^t)$ for all t. It is easy to see that, for $\sigma \in \mathcal{G}(g, \psi)$, $g \in G(\mathbf{A}_f)$, and $t \in \mathbf{A}^{f,\times}$, we have

$$g^t(\sigma) = (j^t)^{-1}(g(j^t(\sigma))) \tag{5.0.1}$$

The equality (5.0.1) and the considerations in Remark 4.10 provide a natural identification of the projective representations $g \mapsto \mathbb{P}(\omega_\psi(g^t))$ and $g \in \mathbb{P}(\omega_{\psi^t}(g))$. In particular, the action of $T(\mathbf{A}_f)$ on $G(\mathbf{A}_f)$ by conjugation lifts to a canonical action on $\tilde{G}(\mathbf{A}_f)$. Let $\tilde{G}Sp(\mathbf{A}_f)$ be the semi-direct product of $T(\mathbf{A}_f)$ with $\tilde{G}(\mathbf{A}_f)$, defined in terms of this action. Then $\tilde{G}Sp(\mathbf{A}_f)$ is am extension of $GSp(\mathbf{A}_f)$ by $\mathbb{G}_m$ in the category of algebraic group schemes over $\mathbb{Q}$. Cf. [PSGe], where something similar is worked out in the case g = 1; the general case is worked out by Vigneras in [V], using explicit cocycles.

The $\tilde{G}(\mathbf{A}_f)$-bundle Θ_ψ over the connected component $\mathcal{M}^g_\psi$ of $\mathcal{M}^g$ apparently does not extend to a bundle over $\mathcal{M}^g$ homogeneous under $\tilde{G}Sp(\mathbf{A}_f)$, essentially for the same reason that the representation $\omega_p si$ does not extend to the larger group. In order to get around this latter obstacle, a number of authors replace $\omega_p si$ by the induced representation from $\tilde{G}(\mathbf{A}_f)$ to $\tilde{G}Sp(\mathbf{A}_f)$. In our setting the natural analogue to the induced representation seems to be a certain $\mathbb{Q}^\times$ torsor over $\mathcal{M}^g$, which we now construct.

We begin by noting that $\mathcal{M}^g$ is the Shimura variety attached to the pair $(GSp, \mathfrak{S}^\pm)$, where $\mathfrak{S}^\pm$ is the union of the Siegel upper and lower half planes, considered as a homogeneous space under $GSp(\mathbb{R})$. In Deligne's formulation of Shimura's theory of canonical models, recalled briefly in §7, below, $\mathfrak{S}^\pm$ is a $GSp(\mathbb{R})$-conjugacy class of homomorphisms of the real torus $\underline{S} \stackrel{\text{def}}{=} R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ into $GSp_\mathbb{R}$ which satisfies certain axioms.

5.1. When $g = 0$, $GSp = \mathbb{G}_m$ and $\mathfrak{S}^\pm$ is just the norm homomorphism $N : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{R}}$. The corresponding Shimura variety, denoted $M(\mathbb{G}_m, N)$, is the profinite scheme

$$\varprojlim_N \mathbb{R}^\times\mathbb{Q}^\times\backslash\mathbf{A}^\times/(1 + N\hat{\mathbb{Z}}) \simeq \mathbb{R}^\times\mathbb{Q}^\times\backslash\mathbf{A}^\times,$$

all of whose points are rational over $\mathbb{Q}^{ab}$. The $\mathbb{Q}$-structure on $M(\mathbb{G}_m, N)$ is defined by the obvious action of $Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^\times$ on $\mathbb{R}^\times\mathbb{Q}^\times\backslash\mathbf{A}^\times$: for $e \in \hat{\mathbb{Z}}^\times$, $a \in$

$\mathbb{R}^\times\mathbb{Q}^\times\backslash\mathbf{A}^\times$, we have $e(a) = e^{-1}\cdot a$. (Here the isomorphism of $Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ with $\hat{\mathbb{Z}}^\times$ is given by the inverse of the cyclotomic character, as in (1.1).) Then $\mathbf{A}^{f,\times}$ acts naturally and $\mathbb{Q}$-rationally on $M(\mathbb{G}_m, N)$.

Any abstract group Δ defines a constant group scheme $Spec(\mathbb{Z}_\Delta)$ over $Spec(\Delta)$, where $\mathbb{Z}_\Delta$ is the direct sum of as many copies of $\mathbb{Z}$ as there are elements in Δ. (This is a special case of the construction in the appendix to §4.) Let $\Delta = \mathbb{Q}^\times$, and let $\tilde{M}(\mathbb{G}_m, N)$ be $\mathbf{A}^{f,\times}$, viewed as a $\mathbb{Q}^\times$-torsor over $M(\mathbb{G}_m, N)$. The action of $Gal(\mathbb{Q}^{ab}/\mathbb{Q})$ on $M(\mathbb{G}_m, N)$, defined above, lifts in the obvious way to an action on $\tilde{M}(\mathbb{G}_m, N)$, which thus becomes a $\mathbb{Q}$-rational $\mathbb{Q}^\times$-torsor over $M(\mathbb{G}_m, N)$. The action of $\mathbf{A}^{f,\times}$ on $M(\mathbb{G}_m, N)$ obviously lifts to $\tilde{M}(\mathbb{G}_m, N)$.

We write M, $\tilde{M}$ instead of $M(\mathbb{G}_m, N)$, $\tilde{M}(\mathbb{G}_m, N)$, respectively. Let M^2 be the quotient of $\tilde{M}$ by the action of $\Delta = \mathbb{Q}_+^\times \subset \mathbb{Q}^\times$. Then M^2 is a double cover of M, defined over $\mathbb{Q}$.

The Shimura variety M has a modular interpretation analogous to that of $\mathcal{M}^g$. We let $T^f(\mathbb{G}_m)$ be the product over all primes ℓ of the ℓ-adic Tate modules of the commutative group scheme $\mathbb{G}_m$, $V^f(\mathbb{G}_m) = T^f(\mathbb{G}_m)\otimes\mathbb{Q}$. Then M parametrizes "isogeny classes" of isomorphisms $\alpha : \mathbf{A}_f \xrightarrow{\sim} V^f(\mathbb{G}_m)$, where α, α' are isogenous if $\alpha(x) \cong \alpha'(a\cdot x)$, for some $a \in \mathbb{Q}^\times$. Clearly, $\tilde{M}$ parametrizes isomorphisms $\alpha : \mathbf{A}_f \xrightarrow{\sim} V^f(\mathbb{G}_m)$, and the morphism $\tilde{M} \to M$ just takes α to its isogeny class. The actions of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\tilde{M}$ and M derive from this modular interpretation. The natural action of $\mathbf{A}^{f,\times}$ on $\tilde{M}$ is given by $\alpha \mapsto (x \mapsto \alpha(t^{-1}x) \overset{\text{def}}{=} \alpha^t(x))$.

Note that α is uniquely determined by the composition

$$\mathbf{A}_f \xrightarrow{\sim} V^f(\mathbb{G}_m) \to V^f(\mathbb{G}_m)/T^f(\mathbb{G}_m) \simeq \mu_\infty.$$

If we denote this composition $\psi = \psi_\alpha$, then for each $\alpha \in \tilde{M}$ we may construct the group $\mathcal{G}(g, \psi) = \mathcal{G}(g, \alpha)$ with multiplication law (3.6); the hypothesis that ψ is trivial on $\hat{\mathbb{Z}}$ is irrelevant. Clearly, $\mathcal{G}(g, \alpha)$ is the fiber at α of a group scheme $\mathcal{G}(g)$ over $\tilde{M}$, defined over $\mathbb{Q}$. We note that the actions of $\mathbf{A}^{f,\times}$ on additive characters and on $\tilde{M}$ correspond: we have $\psi_{(\alpha^t)} = (\psi_\alpha)^t$, in the notation of 4.10.

The action of $\mathbf{A}^{f,\times}$ on $\tilde{M}$ lifts to an action on $\mathcal{G}(g)$, as follows: if $(\beta, x, y)(\alpha) \in \mathcal{G}(g, \alpha)$, $t \in \mathbf{A}^{f,\times}$, then let

$$t\cdot(\beta, x, y)(\alpha) = (\beta, x, ty)(\alpha^t) \tag{5.1.1}$$

The map (5.1.1) is an isomorphism $r(t) : \mathcal{G}(g, \alpha) \xrightarrow{\sim} \mathcal{G}(g, \alpha^t)$.

Finally, M^2 parametrizes isomorphisms $\mathbf{A}_f \xrightarrow{\sim} V^f(\mathbb{G}_m)$ up to multiplication by an element of Δ. The natural map $\tilde{M} \to M^2$ obviously defines an isomorphism from the subscheme

$$\tilde{M}^2 \overset{\text{def}}{=} \{\alpha \in \tilde{M} \mid \ \psi_\alpha : \mathbf{A}_f \to \mu_\infty \text{ is trivial on } \hat{\mathbb{Z}}\} \subset \tilde{M} \tag{5.1.2}$$

onto M^2. We identify M^2 with $\tilde{M}^2$, or alternatively with the set of characters $\mathbf{A}_f/\hat{\mathbb{Z}} \to \mu_\infty$, by means of this isomorphism. The existence of $\tilde{M}^2$ implies that the Δ-torsor $\tilde{M} \to M^2$ is trivial .

5.2. Now let g be arbitrary. One defines in the obvious way a natural homomorphism of group schemes over $\tilde{M}$:

$$\rho : \mathcal{G}(g) \to Aut(\hat{\mathcal{S}}^g_{\tilde{M}}) \tag{5.2.1}$$

such that the fiber at $\alpha \in \tilde{M}$ is just ρ_{ψ_α}. We saw in §4 that $G(\mathbf{A}_f)$ acts naturally on $\mathcal{G}(g, \psi)$, for each ψ, and therefore acts naturally on $\mathcal{G}(g)$, preserving the fibers over $\tilde{M}$. On the other hand, if $\gamma(t) \in T(\mathbf{A}_f)$, we let $\gamma(t)$ act on $\mathcal{G}(g)$ as the morphism $r(t)$ of (5.1.1). The equality (5.0.1) shows that these actions of $G(\mathbf{A}_f)$ and $T(\mathbf{A}_f)$ together define an action of GSp^g on $\mathcal{G}(g)$, covering the action of GSp^g on $\tilde{M} \simeq M^2$ given by $\nu : GSp^g(\mathbf{A}_f) \to \mathbf{A}^{f,\times}$. We have defined our $\mathbb{Q}$-structure on $\tilde{M}$ in such a way as to guarantee that this action of $GSp^g(\mathbf{A}_f)$ on $\mathcal{G}(g)$ is defined over $\mathbb{Q}$.

5.2.2. Similarly, the projective representations $\mathbb{P}(\omega_\psi)$ patch together to define an action $\mathbb{P}(\omega) : G(\mathbf{A}_f) \to Aut(\mathbb{P}(\hat{\mathcal{S}}^g)_{\tilde{M}})$. Again, (5.0.1) implies that this action extends to an action of $GSp^g(\mathbf{A}_f)$ on $\mathbb{P}(\hat{\mathcal{S}}^g)_{\tilde{M}}$, and this action is again defined over $\mathbb{Q}$.

5.3. As above, $GSp^g(\mathbf{A}_f)$ acts on M^2 through $\nu : GSp^g(\mathbf{A}_f) \to \mathbf{A}^{f,\times}$, and there is a natural $GSp^g(\mathbf{A}_f)$-equivariant map $\nu' : \mathcal{M}^g \to M^2$, taking the connected component $\mathcal{M}^g_\psi$ to the character ψ. Composing with the canonical morphism $\hat{\mathcal{A}}^g \to \mathcal{M}^g$, we obtain a $GSp^g(\mathbf{A}_f)$-equivariant map $\hat{\mathcal{A}}^g \to M^2$. We let $\tilde{\mathcal{M}}^g = \mathcal{M}^g \times_{M^2} \tilde{M}$, $\tilde{\mathcal{A}}^g = \hat{\mathcal{A}}^g \times_{M^2} \tilde{M}$. Then $\tilde{\mathcal{M}}^g$ (resp. $\tilde{\mathcal{A}}^g$) is a $\mathbb{Q}$-rational Δ-torsor over $\mathcal{M}^g$ (resp. $\hat{\mathcal{A}}^g$), and the actions of $GSp^g(\mathbf{A}_f)$ on $\mathcal{M}^g$ and $\hat{\mathcal{A}}^g$ lift to actions on $\tilde{\mathcal{M}}^g$ and $\tilde{\mathcal{A}}^g$, which commute with the action of Δ. The identity section $\mathcal{M}^g \to \tilde{\mathcal{A}}^g$ is denoted ε, as in §4.

Now the inclusion $\mathcal{M}^g_\psi \subset \mathcal{M}^g$, together with the constant map from $\mathcal{M}^g_\psi$ to the point $\psi \in \tilde{M}^2 \subset \tilde{M}$, define a $G(\mathbf{A}_f)$-equivariant morphism $\mathcal{M}^g_\psi \to \tilde{\mathcal{M}}^g$. Similarly, we have a $G(\mathbf{A}_f)$-equivariant map $\hat{\mathcal{A}}^g_\psi \to \tilde{\mathcal{A}}^g$.

5.4. The representation ρ of (5.2.1) provides us with a ($\mathbb{Q}$-rational) morphism of schemes over $\tilde{M}$ generalizing Theorem 4.7;

$$\vartheta : \tilde{\mathcal{A}}^g \to \mathbb{P}(\hat{S}^g)_{\tilde{M}}.$$

The fiber of ϑ at the point $\psi \in \tilde{M}$ is just ϑ_ψ. We let $\Theta(\mathcal{A}) = \vartheta^*(\mathcal{O}_{\mathbb{P}(\hat{\mathcal{S}}^g)_{\tilde{M}}})$, $\Theta = \varepsilon^*(\Theta(\mathcal{A}))$. Then Θ (resp. $\Theta(\mathcal{A})$ is a $\mathbb{Q}$-rational vector bundle over $\tilde{\mathcal{M}}^g$, (resp. $\tilde{\mathcal{A}}^g$) whose restriction to $\mathcal{M}^g_\psi$ is Θ_ψ. The construction in 5.2.2 shows that the action of $\tilde{G}(\mathbf{A}_f)$ on Θ_ψ , defined by Proposition 4.9, extends to a $\mathbb{Q}$-rational action of $\tilde{G}Sp(\mathbf{A}_f)$ on Θ, the pullback via ε of a $\mathbb{Q}$-rational action on $\Theta(\mathcal{A})$. We have proved:

Proposition 5.5. *There is a $\mathbb{Q}$-rational, $\tilde{G}Sp(\mathbf{A}_f)$ -equivariant line bundle Θ over the scheme $\tilde{\mathcal{M}}^g$. The restriction of Θ to the subscheme $\mathcal{M}^g_\psi \subset \tilde{\mathcal{M}}^g$ is $\tilde{G}(\mathbf{A}_f)$-equivariantly isomorphic to Θ_ψ.*

5.6. For $n = 0, 1, \ldots$, let $J^n(\Theta)$ denote the bundle of n-jets of $\Theta(\mathcal{A})$; let

$$J^\infty(\Theta) = \varprojlim_n J^n(\Theta).$$

The action of $\tilde{G}Sp(\mathbf{A}_f)$ on $\Theta(\mathcal{A})$ defines an action on $J^\infty(\Theta)$. Any global section s of $\Theta(\mathcal{A})$ defines a global section $j^\infty(s)$ of $J^\infty(\Theta)$.

Let $\mathcal{D} = \mathcal{D}_{\tilde{\mathcal{A}}^g}$ denote the $\mathcal{O} = \mathcal{O}_{\tilde{\mathcal{A}}^g}$-algebra of finite-order algebraic differential operators on $\tilde{\mathcal{A}}^g$. It follows from the definitions (cf. [H,§7]) that

$$J^\infty(\Theta) \simeq (\mathcal{D} \otimes_{\mathcal{O}} \Theta^*)^*, \tag{5.6.1}$$

where we regard $\mathcal{D}$ as a right $\mathcal{O}$-module and Θ^* as a left $\mathcal{O}$-module. Here the right-hand side is viewed as an inverse limit of coherent sheaves, defined by the order filtation on $\mathcal{D}$.

The isomorphism (5.6.1) makes $J^\infty(\Theta)$ naturally into a right $\mathcal{D}$-module: if $f \in \Gamma(U, J^\infty(\Theta)) \simeq \Gamma(U, Hom(\mathcal{D} \otimes_{\mathcal{O}} \Theta^*, \mathcal{O}))$, and $\Delta \in \Gamma(U, \mathcal{D})$, for U open in $\tilde{\mathcal{A}}^g$, then

$$f \star \Delta(g) = f(\Delta \cdot g),\ \forall g \in \Gamma(U, \mathcal{D} \otimes \mathcal{O}\Theta^*).$$

Let $\mathcal{E}^\infty(\Theta)$ denote the $\mathcal{O}_{\tilde{\mathcal{A}}^g}$ subbundle of $J^\infty(\Theta)$ generated by $(j^\infty(\Gamma(\tilde{\mathcal{A}}^g, \Theta(\mathcal{A}))) \star \mathcal{D}$, and let $\mathcal{S}^\infty(\Theta) = \varepsilon^*(\mathcal{E}^\infty(\Theta))$. Then the action of $\tilde{G}Sp(\mathbf{A}_f)$ on $\tilde{\mathcal{M}}^g$ lifts to an action on $\mathcal{S}^\infty(\Theta)$. The bundle $\mathcal{S}^\infty(\Theta)$, together with its natural $\tilde{G}Sp(\mathbf{A}_f)$-action, will be the subject of §7 and §8.

6. Relations with the analytic theory

In order to apply the theory developed in the preceding sections to the arithmetic of the oscillator representation, we must describe the relations between the vector bundles Θ and the analytic theory of theta functions. This material is in principle well known, having been covered by numerous authors, including [I, M2, M3]. Our formulation is somewhat different, however, from those currently available in the literature. Most of this section will therefore be taken up with definitions; proofs will be brief.

6.0. Notation. In this section, $\psi_{\mathbf{A}} : \mathbb{Q}\backslash\mathbf{A} \to \mathbb{C}^\times$ will be a continuous character; we denote by ψ (resp. ψ_∞) the restriction of $\psi_{\mathbf{A}}$ to $\mathbf{A}_f$ (resp to $\mathbb{R}$). We define the group $\mathcal{G}(g, \psi_{\mathbf{A}})$ (resp. $\mathcal{G}(g, \psi_\infty)$) to be the set $\mathbb{C}^\times \times \mathbf{A}^g \times \mathbf{A}^g$ (resp. $\mathbb{C}^\times \times \mathbb{R}^g \times \mathbb{R}^g$) with multiplication law (3.6), where ψ is replaced by $\psi_{\mathbf{A}}$ (resp. ψ). Note that ψ is necessarily of the form $\psi(x) = e^{2\pi\lambda x}, x \in \mathbb{R}$, for some $\lambda \in \mathbb{R}$. We define $\psi_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^\times$ by the formula $\psi_{\mathbb{C}}(z) = e^{2\pi\lambda z}, z \in \mathbb{C}$, and define $\mathcal{G}(g, \psi_{\mathbb{C}})$ (resp. $\mathcal{G}(g, \psi_{\mathbb{C}} \cdot \psi)$ to be the set $\mathbb{C}^\times \times \mathbb{C}^g \times \mathbb{C}^g$ (resp. $\mathbb{C}^\times \times (\mathbb{C}^\times \mathbf{A}_f)^g \times (\mathbb{C}^\times \mathbf{A}_f)^g$) with multiplication law (3.6), where ψ is replaced by $\psi_{\mathbb{C}}$ (resp. $\psi_{\mathbb{C}} \times \psi$). Then $\mathcal{G}(g, \psi_{\mathbb{C}}$ is a complex Lie group; and $Lie(\mathcal{G}(g, \psi_\infty))$ is a Lie subalgebra of $Lie(\mathcal{G}(g, \psi_{\mathbb{C}}))$. When necessary, we let $Z(\mathcal{G}) = \mathbb{C}^\times \subset \mathcal{G}(g, \star)$, $\star = \psi$, $\psi_{\mathbf{A}}$, ψ_∞, $\psi_{\mathbb{C}}$, or $\psi_{\mathbb{C}} \cdot \psi$. Note that there is an involution $D_{-1} : \mathcal{G}(g, \star) \to \mathcal{G}(g, \star)$, defined as in §3, with any $\star$ as above.

We let $\underline{S}$ be the real algebraic group $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. A *Hodge structure* on a real vector space V is a homomorphism $h : \underline{S} \to GL(V)$ of real algebraic groups. Then h defines a Hodge decomposition and a Hodge filtration

$$V_{\mathbb{C}} \bigoplus_{p,q} V^{p,q},\ F^pV = \bigoplus_{p' \geq p} V^{p',q},$$

where $\underline{S}_{\mathbb{C}}$ acts on $V^{p,q}$ through the character $z^{-p}\bar{z}^{-q}$. The Hodge structure h is a complex structure if and only if $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$.

Let $V = \mathbb{Q}^{2g}$ as in 1.1; thus $\mathcal{G}(g,\psi)$ is isomorphic as a manifold to $V(\mathbb{R}) \times \mathbb{C}^\times$. Let $<,>$ and $GSp = GSp(V,<,>)$ be defined as in §5. Let

$\mathfrak{S}^{\pm} = \{h : \underline{\ \ \rightarrow} GSp(\mathbb{R}) \mid h$ defines a complex structure on $V(\mathbb{R})$ and $(x,y)_h \stackrel{def}{=} < x, h(i)y >$ is a positive- or negative-definite symmetric form on $V(\mathbb{R})\}$.

Then conjugation by $GSp(\mathbb{R})$ makes $\mathfrak{S}^{\pm}$ into a homogeneous space for $GSp(\mathbb{R})$, and it is well known that $\mathfrak{S}_{\pm}$ is naturally isomorphic to the union of the Siegel upper and lower half-planes, introduced in §5. In particular, our notation is consistent. We let $\mathfrak{S}^{+}$ be the subset of $\mathfrak{S}^{\pm}$ for which $(x,y)_h$ is positive-definite.

6.1. We will be concerned with the following situation. Let $H \subset G$ be real Lie groups such that G/H has a G-invariant complex structure. (In practice, G will be an adelic group, but the reduction to this case is easy.) The group H is assumed to be reductive, but G is not, nor is G even assumed to be algebraic. We write $\mathfrak{g} = Lie(G)$, $\mathfrak{h} = Lie(H)$, and

$$G_{\mathbb{C}} = H = \mathbb{C} \oplus \mathfrak{q}^{+} \oplus \mathfrak{q}^{-},$$

where $\mathfrak{q}^{+}$ (resp. $\mathfrak{q}^{-}$) corresponds to the holomorphic (resp. anti-holomorphic) tangent space at the identity coset in G/H.

Let $G_{\mathbb{C}}$ denote a complex analytic Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, $H_{\mathbb{C}}$ the subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{h}_{\mathbb{C}}$; we assume $\mathfrak{q}^{+}$ and $\mathfrak{q}^{-}$ to be the Lie algebras of commutative subgroups Q^{+} and Q^{-}, respectively, of $G_{\mathbb{C}}$, each normalized by $H_{\mathbb{C}}$. We assume there is a homomorphism $G \rightarrow G_{\mathbb{C}}$ with finite kernel, contained in H; let G' (resp. H') be the image of G (resp. H) under this map. Let $\mathcal{Q} = H_{\mathbb{C}} \cdot Q^{-}$; we assume $\mathcal{Q} \cap G' = H'$ and that the complex structure on G/H is induced by the natural open immersion $G/H \hookrightarrow G_{\mathbb{C}}/\mathcal{Q}$.

We let $\Gamma \subset G$ be a discrete subgroup which acts properly discontinuously on G/H, and we let $M = \Gamma\backslash G/H$. A representation $\rho : H \rightarrow GL(V_\rho)$ determines a C^∞ vector bundle $[\rho]$ on M:

$$[\rho] = \Gamma\backslash G \times V_\rho/H,$$

where Γ acts trivially on V_ρ and H acts on $G \times V_\rho$ by the formula

$$(g,v) \cdot h = (gh, \rho(h)^{-1}v), g \in G, v \in V_\rho, h \in H. \tag{6.1.1}$$

In general V_ρ need not be finite-dimensional, but it should be either a direct sum or an inverse limit of finite-dimensional H-modules. If U is an open subset of M and $\mathcal{E}$ is a C^∞ vector bundle over M, we denote by $\Gamma^\infty(U,\mathcal{E})$ the space of C^∞ sections of $\mathcal{E}$ over U. In the cases to be considered below, the bundle $[\rho]$ will have a holomorphic structure defined as follows. Let $p : \Gamma\backslash G \rightarrow M$ be the natural projection. For any open $U \subset M$, there is a natural isomorphism

(6.1.2)
$$\underline{Lift} : \Gamma^\infty(U,[\rho]) \xrightarrow{\sim} \{f \in C^\infty(p^{-1}(U)) \mid f(gk) = \rho^{-1}(k)f(g), \forall g \in p^{-1}(U), k \in H\}.$$

We denote the right hand side by $C^\infty(U,\rho)$, and write $C^\infty(\rho) = C^\infty(M,\rho)$. The holomorphic structure on $[\rho]$ will be given by the sheaf associated to the presheaf

$$\mathcal{H}(U,\rho) = \{s \in \Gamma^\infty(U,[\rho]) \mid X \cdot \underline{Lift}(s) \equiv 0, \forall X \in \mathfrak{q}^{-}\}. \tag{6.1.3}$$

Here $\mathfrak{q}^-$ acts by right differentiation on $C^\infty(G)$. Again, let $\mathcal{H}(\rho) = \mathcal{H}(M,\rho)$.

We assume our space M to be endowed with an algebraic structure compatible with the holomorphic structure defined above, and assume $[\rho]$ to be an algebraic vector bundle. We let $\mathcal{D}^{an}$ (resp. $\mathcal{D}$) denote the sheaf of analytic (resp. algebraic) differential operators on M; let $\mathcal{O}^{an}$ (resp. $\mathcal{O}$) denote the structure sheaf of M as an analytic (resp. algebraic) variety. Let U (resp. U') be the enveloping algebra of $G_{\mathbb{C}}$ (resp. $H_{\mathbb{C}} \oplus \mathfrak{q}^-$). Now the vector bundle of analytic differential operators on $G_{\mathbb{C}}/\mathcal{Q}$ is naturally isomorphic to

$$G_{\mathbb{C}} \times (U \otimes_{U'} \mathbb{C})/\mathcal{Q},$$

where $\mathbb{C}$ is the trivial U'-module and the action of H on U is given by the adjoint representation. There is thus an isomorphism of analytic vector bundles

$$\mathcal{D}^{an} \simeq \Gamma\backslash G \times U \otimes_{U'} \mathbb{C}/H = [\rho_U],$$

where ρ_U is the adjoint representation of H on $U \otimes_{U'} \mathbb{C}$. More generally, we have

$$\mathcal{D}^{an} \otimes_{\mathcal{O}^{an}} [\rho]^a n \simeq \Gamma\backslash G \times U \otimes_{U'} V_\rho/H,$$

where the action of $\mathfrak{h}_{\mathbb{C}}$, extended trivially to $\mathfrak{q}^-$, makes V_ρ naturally into a U'-module and where H acts diagonally on $U \otimes_{U'} V_\rho$.

Here and in (6.1.6) below, the holomorphic structure is defined by a variant of (6.1.3): $d\rho$ is a representation of $\mathfrak{h}_{\mathbb{C}} oplus \mathfrak{q}^-$ (which does not necessarily integrate to a representation of $\mathcal{Q}$), and the holomorphic sections of $[\rho]$ over an open set U are those which lift to $\mathfrak{q}^-$-invariant V_ρ-valued functions on $p^{-1}(U)$. For example, in (6.1.4) $d\rho$ is the adjoint representation. The same construction works for any finite dimensional $H_{\mathbb{C}} \oplus \mathfrak{q}^-$-module V_ρ.

Of course, $\mathcal{D}$ has a filtration $\mathcal{O} = \mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}_i \subset \dots$ by degree of differential operators. The enveloping algebra $U(\mathfrak{q}^+)$ is isomorphic to the symmetric algebra $S(\mathfrak{q}^+)$; we denote by $S_i(\mathfrak{q}^+) \subset S(\mathfrak{q}^+)$ the subspace of tensors of degree $\leq i$; then $\mathcal{D}_i^{an} \simeq \Gamma\backslash G \times S_i(\mathfrak{q}^+)/H$.

Let $J^n[\rho]$ (resp. $J^\infty[\rho]$) denote the bundle of n-jets (resp. the bundle $\varprojlim_n J^n[\rho]$ of infinite jets) of $[\rho]$. As in (5.6.1), we have

$$J^n[\rho] \simeq (\mathcal{D}_n \otimes_{\mathcal{O}} [\rho]^\star)^\star,\ J^\infty[\rho] \simeq (\mathcal{D} \otimes_{\mathcal{O}} [\rho]^\star)^\star \tag{6.1.5}$$

and $J^\infty[\rho]$ is a right $\mathcal{D}$-module. In the category of analytic vector bundles, there is thus an isomorphism

$$J^\infty[\rho] \simeq \Gamma\backslash G \times (U \otimes_{U'} V_\rho^\star)^\star/H \tag{6.1.6}$$

where the dual $(U \otimes_{U'} V_\rho^\star)^\star$ is regarded as $\varprojlim(S_i(\mathfrak{q}^+) \otimes V_\rho^\star)^\star$. The isomorphisms of C^∞ vector bundles:

$$S_i(\mathfrak{q}^+) = \bigoplus_{i=0}^n Sym^i(\mathfrak{q}^+)$$

defines a decomposition of C^∞ vector bundles

$$\begin{aligned} J^n[\rho] &\simeq \bigoplus_{i=0}^n \Gamma\backslash G \times Hom(Sym^i(\mathfrak{q}^+), V_\rho)/H \\ &\simeq \bigoplus_{i=0}^n Sym^i \Omega^1_M \otimes [\rho]. \end{aligned} \tag{6.1.7}$$

Here H acts by the adjoint action on $Sym^i(\mathfrak{q}^+)$ and by ρ on V_ρ. Note that the lower isomorphism is not holomorphic. Let $\rho(i)$ denote the natural action of H on $Hom(Sym^i(\mathfrak{q}^+), V_\rho)$. We have analogously an isomorphism of C^∞ vector bundles

$$\mathcal{D}^{an} \simeq \bigoplus_{i=0}^{n} \Gamma\backslash G \times Sym^i(\mathfrak{q}^+)/H \simeq \bigoplus_{i=0}^{\infty} Sym^i T_M \tag{6.1.8}$$

where Hacts trivially on $Sym^i(\mathfrak{q}^+)$ and T_M is the tangent bundle to M. In terms of (6.1.7.8), the right action of $\mathcal{D}^{an}$ on $J'[\rho]$ is given by the natural pairing

$$\bigoplus_{i=0}^{\infty} Sym^i T_M \otimes \hat{\bigoplus}_{i=0}^{\infty} Sym^i \Omega^1_M \otimes [\rho] \tag{6.1.9}$$

defined by contraction of tangent and cotangent vectors; here $\hat{\bigoplus}$ denotes the completed direct sum.

6.1.10. We let $j^n : [\rho] \to J^n[\rho]$ denote the differential operator which takes a section of $[\rho]$ to its n-jet, $n \leq \infty$. Let $s \in \Gamma^\infty(M, [\rho])$, and let $\mathcal{E}^\infty(s)$ denote the $\mathcal{O}$-subbundle of $J^\infty[\rho]$ generated by $j^\infty(s) \star \mathcal{D}$. Let $p : \Gamma\backslash G \ \to \ M$ be the projection, as above. The fiber of $\mathcal{E}^\infty(s)$ at the point $p(g)$ is canonically isomorphic to the subspace of $Hom(U, V_\rho)$ given by

$$D \mapsto r(X) D \underline{Lift}(s)(g), \qquad X \in U.$$

Here $X \mapsto r(X)$ denotes the right action of U on $C^\infty(G)$. If we denote by $V(s)$ the U-submodule $r(U)\underline{Lift}(s) \subset C^\infty(\Gamma\backslash G)$, then the fiber of $\mathcal{E}^\infty(s)$ at $p(g)$ is naturally a U'-submodule of $Hom(V(s), V_\rho)$. Of course, if s is holomorphic, then $V(s)$ is a quotient of the "generalized Verma module" $U \otimes_{U'} V_\rho^\star$

Write $\omega = \Omega^1_M$. and let $\delta_i : [\rho] \to Sym^i \omega \otimes [\rho]$, $i = 0, \ldots, n$ be the C^∞ differential operator obtained as the composition of j^n with the projection on the i-th factor in (6.1.9). Let $\Delta^i : C^\infty(\rho) \to C^\infty(\rho^{(i)})$ be the homomorphism which makes the following diagram commute:

$$\begin{array}{ccc} \Gamma^\infty([\rho]) & \xrightarrow[\underline{Lift}]{\sim} & C^\infty(\rho) \\ \Big\downarrow \delta^i & & \Big\downarrow \Delta^i \\ \Gamma^\infty(Sym^i \omega \otimes [\rho]) & \xrightarrow[\underline{Lift}]{\sim} & C^\infty(\rho^{(i)}) \end{array}$$

If $f \in C^\infty(\rho)$, we have

$$\Delta^i(f)(X) = X \cdot f \in C^\infty(G, V_\rho) \ \forall X \in Sym^i(\mathfrak{q}^+) \tag{6.1.11}$$

6.2. In this section, we apply the above theory when $G = \mathcal{G}(g, \psi_{\mathbf{A}})$, $H = Z(\mathcal{G})$ (notation 6.0). Since ψ is trivial on $\mathbb{Q}$, the set $H(\mathbb{Q}) \stackrel{def}{=} 1 \times \mathbb{Q}^g \times \mathbb{Q}^g \subset \mathcal{G}(g, \psi_{\mathbf{A}})$ is a subgroup, which serves as our Γ. We let $\rho(-1)$ be the character $t \ \mapsto \ t^{-1}$ of $Z(\mathcal{G})$, and write

$$\hat{T}_\psi = H(\mathbb{Q})\backslash \mathcal{G}(g, \psi_{\mathbf{A}})/\mathbb{C}^\times \ (= M, \text{ as above}),\ \hat{L}_\psi = [\rho(-1)] \tag{6.2.1}$$

Let $K \subset (\mathbf{A}_f)^g \times (\mathbf{A}_f)^g$ be an open compact subgroup such that $\{1\} \times K \subset \mathbb{C}^\times \times (\mathbf{A}_f)^g \times (\mathbf{A}_f)^g \simeq \mathcal{G}(g,\psi)$ is a subgroup of $\mathcal{G}(g,\psi)$. Then we may identify K with the subgroup $\{1\} \times K \subset \mathcal{G}(g,\psi)$. Let $T_\psi(K) = \hat{T}_\psi/K$, $L_\psi(K) = \hat{L}_\psi/K$. Then $T_\psi(K)$ is isomorphic to the compact torus $\mathbb{R}^2 g/K \cap \mathbb{Q}^{2g}$ and $L_\psi(K)$ is a complex line bundle over $T_\psi(K)$. Any $h \in \mathfrak{S}^+$ defines a complex structure on $\mathbb{R}^2 g$ and hence on the torus $T_\psi(K)$; it is well known that $T_\psi(K)$, with this complex structure, is isomorphic to an abelian variety, which we denote $A_{h,\psi}(K)$. We let $\hat{A}_{h,\psi}$ be the isocompletion of $A_{h,\psi}(K)$, for any K. Then $\hat{A}_{h,\psi}$ does not depend on K and is topologically isomorphic to T_ψ. Moreover, the isomorphism $\hat{A}_{h,\psi} \xrightarrow{\sim} T_\psi$ defines a full level structure $\hat{A}_{h,\psi}(t) \xrightarrow{\sim} V \otimes \mathbf{A}_f$, and is thus an isomorphism of topological spaces with $\mathcal{G}(g,\psi)$-action.

We let $F_h^\bullet(V)$ denote the Hodge filtration (6.0) on $V_\mathbb{C}$ defined by h. Then $\mathcal{Q}_h \overset{def}{=} F_h^0(V) \times \mathbb{C}^\times$ is an analytic subgroup of $\mathcal{G}(g,\psi_\mathbb{C}\cdot\psi)$, and we have naturally

$$\hat{A}_{h,\psi} \simeq H(\mathbb{Q})\backslash\mathcal{G}(g,\psi_\mathbb{C}\cdot\psi)/\mathcal{Q}_h,\ \hat{L}_\psi = H(\mathbb{Q})\backslash\mathcal{G}(g,\psi_\mathbb{C}\cdot\psi)\times\mathbb{C}/\mathcal{Q}_h. \tag{6.2.2}$$

Here the subgroup $F_h^0(V)$ of $\mathcal{Q}_h$ acts trivially on $\mathbb{C}$. This shows that $\hat{L}_\psi$ is naturally a holomorphic vector bundle over $\hat{A}_{h,\psi}$; with this holomorphic structure we write $\hat{L}_{h,\psi}$ instead of $\hat{L}_\psi$. In fact, it is well known that $\hat{L}_{h,\psi}$ is even algebraic.

In the notation of 6.1, we have $G_\mathbb{C} = \mathcal{G}(g,\psi_\mathbb{C}\cdot\psi)$, $\mathcal{Q} = \mathcal{Q}_h$. We write $\mathfrak{u}_h^+ = V^{-1,0}$, $\mathfrak{u}_h^- = V^{0,-1}$, instead of $\mathfrak{q}^+$, $\mathfrak{q}^-$. We write $\mathcal{H}(\hat{L}_\psi)$ for $\mathcal{H}(\rho(-1))$ (6.1.3), and define $\underline{Lift} : \Gamma(\hat{A}_{h,\psi}, \hat{L}_\psi) \xrightarrow{\sim} \mathcal{H}(\rho(-1))$ as in (6.1.2).

We define the Schwartz space $\mathcal{S}(\mathbb{R}^g)$ in the usual way (cf. e.g., []), and let $\mathcal{S}(\mathbf{A}^g) = \mathcal{S}(\mathbb{R}^g) \otimes \mathcal{S}^g$, with $\mathcal{S}^g$ as in §3. Then $\mathcal{G}(g,\psi_\mathbf{A})$ acts on $\mathcal{S}(\mathbf{A}^g)$:

$$\rho\psi_\mathbf{A}(\alpha,x,\ell)\Phi(y) = \alpha\cdot\psi_\mathbf{A}({}^t\ell\cdot y)\Phi(x+y), \alpha\in\mathbb{C}^\times, x,\ell,y\in\mathbf{A}^g. \tag{6.2.4}$$

We let $d\rho_{\psi_\mathbb{C}} : Lie(\mathcal{G}(g,\psi_\mathbb{C})) \to End(\mathcal{S}(\mathbb{R}^g))$ denote the corresponding Lie algebra action; the action of $Lie(\mathcal{G}(g,\psi_\mathbb{C}))$ on $\mathcal{S}(\mathbf{A}^g)$ is also denoted $d\rho_{\psi_\mathbb{C}}$. The following lemma is well known (cf. [,]):

Lemma 6.2.5. . *For any h, there is a unique function $\Phi_h \in \mathcal{S}(\mathbb{R}^g)$ such that (i) $d\rho_{\psi_\mathbb{C}}(\mathfrak{u}_h^-)\Phi_h = 0$, and (ii) $\Phi_h(0) = 1$.*

Now for any $\Phi \in \mathcal{S}(\mathbf{A}^g)$, the series

$$\Theta_{\psi_\mathbf{A}}(\Phi)(g) = \sum_{\xi\in\mathbb{Q}^g} \rho_{\psi_\mathbf{A}}(g)\Phi(\xi),\ g\in\mathcal{G}(g,\psi_\mathbf{A}) \tag{6.2.6}$$

converges absolutely and uniformly on compact subsets to a C^∞ function on $\mathcal{G}(g,\psi_\mathbf{A})$ ([], p.), which satisfies

$$\Theta_{\psi_\mathbf{A}}(\Phi)(\gamma g t) = t\Theta_{\psi_\mathbf{A}}(\Phi)(g),\ \forall\gamma\in H(\mathbb{Q}), g\in\mathcal{G}(g,\psi_\mathbf{A}), t\in Z(\mathcal{G}). \tag{6.2.7}$$

In particular, if $Phi = \Phi_h \otimes \Phi^f$, with $\Phi^f \in \mathcal{S}^g$, then $\Theta_{\psi_\mathbf{A}}(\Phi) \in \mathcal{H}(\hat{L}_\psi)$. We thus obtain a homomorphism

$$S_\psi : \mathcal{S}^g \to \Gamma(\hat{A}_{h,\psi}, \hat{L}_\psi); S_\psi(\Phi^f) = \underline{Lift}^{-1}\Theta_{\psi_\mathbf{A}}(\Phi_h\otimes\Phi^f).$$

6.2.8. Let $K \subset (\mathbf{A}_f)^g \times (\mathbf{A}_f)^g$ be an open compact subgroup such that $\{1\} \times K \subset \mathbb{C}^\times \times (\mathbf{A}_f)^g \times (\mathbf{A}_f)^g \simeq \mathcal{G}(g,\psi)$ is a subgroup of $\mathcal{G}(g,\psi)$. Now the involution D_{-1} of $\mathcal{G}(g,\psi_{\mathbb{C}})$, mentioned in 6.0, descends to multiplication by -1 on $A_{h,\psi}(K)$. Then the homomorphism $D_{-1} \times 1 : \mathcal{G}(g,\psi_{\mathbb{C}}) \times \mathbb{C} \to \mathcal{G}(g,\psi_{\mathbb{C}}) \times \mathbb{C}$ induces an isomorphism $L_\psi(K) \xrightarrow{\sim} (-1)^\star L_\psi(K)$. One verifies immediately that $L_\psi(K)$ is a totally symmetric line bundle over $A_{h,\psi}(K)$, for all K. Moreover, if $Z(K)$ is the centralizer in $\mathcal{G}(g,\psi)$ of $\{1\} \times K$, then $Z(K)$ acts on the right on $L_\psi(K)$, and this action defines a canonical isomorphism $Z(K)/K \xrightarrow{\sim} \underline{\mathcal{G}}(L_\psi(K))$ (cf. [M2], p. 237).

We let $\mathcal{T}_{h,\psi}$ be the tower $\{A_{h,\psi}(K)\}, \mathcal{P}_{h,\psi}$ the polarization defined by $\{L_\psi(K)\}$. It follows from the above remarks that the polarized tower $(\mathcal{T}_{h,\psi}, \mathcal{P}_{h,\psi})$ comes equipped with a canonical symmetric θ-structure $\mathcal{G}(\mathcal{P}_{h,\psi}) \xrightarrow{\sim} \mathcal{G}(g,\psi)$. There is thus a canonical action of $\mathcal{G}(g,\psi)$ on $\varinjlim \Gamma(A_{h,\psi}(K), L_\psi(K)) = \Gamma(\hat{A}_{h,\psi}, \hat{L}_\psi)$, given in terms of the realization (6.1.1) by right multiplication. We denote this action ρ'.

Lemma 6.2.9. *The homomorphism $S_\psi : \mathcal{S}^g \to \Gamma(\hat{A}_{h,\psi}, \hat{L}_\psi)$ is an isomorphism, and intertwines the representation ρ_ψ of $\mathcal{G}(g,\psi)$ on $\mathcal{S}^g$ with the action ρ'.*

Proof. Evidently, for any $\Phi^f \in \mathcal{S}^g$, $\Phi = \Phi_h \otimes \Phi^f$, we have $\underline{Lift}(\rho'(h)S_\psi(\Phi))(g) = \Theta_{\psi_{\mathbf{A}}}(\Phi)(gh) = \Theta_{\psi_{\mathbf{A}}}(\rho\psi(h)\Phi)(g)$ by (6.2.6). This implies that $S_\psi(\mathcal{S}^g)$ is a $\mathcal{G}(g,\psi)$-stable subspace of $\Gamma(\hat{A}_{h,\psi}, \hat{L}_\psi)$. Since S_ψ is not identically zero (cf. []) and the action of $\mathcal{G}(\mathcal{P}_{h,\psi})$ on $\Gamma(\hat{A}_{h,\psi}, \hat{L}_\psi)$ is irreducible (cf. [M1], II, p. 109), it follows that S_ψ is an isomorphism.

6.3. Let $\mathcal{O}_{h,\psi}$ denote the structure sheaf of $\hat{A}_{h,\psi}$, and let $\mathcal{D}_{h,\psi}$ denote the $\mathcal{O}_{h,\psi}$-module of finite order algebraic differential operators on $\hat{A}_{h,\psi}$.

We write $C^{\infty,i}(h,\psi)$ for $C^\infty(\rho(-1)^{(i)})$, $i = 0, 1, \ldots,$, (cf. (6.1.10)), and set

$$\Gamma^i(h,\psi)) = \Gamma^\infty(\hat{A}_{h,\psi}, Sym^i \omega_{h,\psi} \hat{L}_\psi)).$$

We define $\underline{Lift} : \Gamma^i(h,\psi) \to C^{\infty,i}(h,\psi)$ as above. Let

$$\Delta_i : C^{\infty,0}(h,\psi) \to C^{\infty,i}(h,\psi)$$

be the homomorphism defined in 6.1.10. The following lemma follows immediately from (6.1.11):

Lemma 6.3.1. *Let $\Phi = \Phi_h \otimes \Phi^f$, with $\Phi^f \in \mathcal{S}^g$. Then*

$$\Delta_i(\Theta_{\psi_{\mathbf{A}}}(\Phi))(X)(g) = \Theta_{\psi_{\mathbf{A}}}(d\rho_{\psi_{\mathbb{C}}}(X)\Phi)(g) \forall g \in \mathcal{G}(g,\psi_{\mathbf{A}}), X \in Sym^i(\mathfrak{u}_h^+),$$

where we regard $Sym^i(\mathfrak{u}_h^+)$ as a subspace of U and where $d\rho_{\psi_{\mathbb{C}}}$ is the action of U on $\mathcal{S}(\mathbb{R}^g)$ determined by (6.2.4).

Corollary 6.3.2. *Let $\mathcal{S}^\infty(\hat{L}_\psi)$ denote the $\mathcal{O}_{h,\psi}$-submodule of $J^\infty(\hat{L}_\psi)$ generated by $(j^\infty(\Gamma(\hat{A}_{h,\psi}, \hat{L}_\psi))) \star \mathcal{D}_{h,\psi}$. Then $\mathcal{S}^\infty(\hat{L}_\psi) = J^\infty(\hat{L}_\psi)$.*

Proof. Let J^n denote the image of $\mathcal{S}^\infty(\hat{L}_\psi)$ in $J^n(\hat{L}_\psi)$, $n = 0, 1, \ldots$. It suffices to show that $J^n = J^n(\hat{L}_\psi)$ for all n. For this we use induction on n. When $n = 0$, this is equivalent to the statement that $\hat{L}_\psi$ is generated by its global sections, i.e., that $\hat{L}_\psi$ defines a polarization.

Now suppose $J^{n-1} = J^{n-1}(\hat{L}_\psi)$. It suffices to show that the global sections of J^n generate $Sym^n\omega_{h,\psi} \otimes \hat{L}_\psi$ at every point, under the map defined by (6.1.7). Let $\Gamma^n(\mathcal{S})$ denote the image of $\Gamma(\hat{A}_{h,\psi}, J^n)$ in $\Gamma^n(h,\psi)$, and let $\mathcal{H}^n(\mathcal{S}) = \underline{Lift}(\Gamma^n(\mathcal{S}))$. It follows from the Lemma and (6.1.9) that

$$\mathcal{H}^n(\mathcal{S}) = \{X \in \Theta_{\psi_{\mathbf{A}}}(d\rho_{\psi_{\mathbb{C}}}(X)\Phi_h \otimes \Phi^f), X \in Sym^n(\mathfrak{u}_h^+) \mid \ \Phi^f \in \mathcal{S}^g\}. \tag{6.3.2.1}$$

Write $\Psi(X,\Phi^f) = \Theta_{\psi_{\mathbf{A}}}(d\rho_{\psi_{\mathbb{C}}}(X)\Phi_h \otimes \Phi^f)$. To complete the induction, we must show that, for every $g \in \mathcal{G}(g,\psi_{\mathbf{A}})$, the homomorphism

$$\begin{aligned} &\mathcal{S}^g \to Sym^n(\mathfrak{u}_h^+)^\star : \\ &\Phi^f \mapsto (X \mapsto \Psi(X,\Phi^f)(g)), X \in Sym^n(\mathfrak{u}_h^+) \end{aligned} \tag{6.3.2.2}$$

is surjective. Suppose not. Then for some non-zero $X \in Sym^n(\mathfrak{u}_h^+)$, we have $\Psi(X,\Phi^f)(g) = 0$ for all Φ^f. It follows that,

$$\Psi(X,\Phi^f)(g\cdot g^f) = 0 \ \forall \Phi^f \in \mathcal{S}^g, \ \forall g^f \in \mathcal{G}(g,\psi).$$

But $\Psi(X,\Phi^f)$ is $H(\mathbb{Q})$-invariant; it follows by continuity that $\Psi(X,\Phi^f)$ is identically zero for all $\Phi^f \in \mathcal{S}^g$, which is absurd. The lemma follows.

6.4. Let $G = Sp(V,<,>)$ as in §1. Then $G(\mathbf{A})$ acts naturally on $\mathcal{G}(g,\psi_{\mathbf{A}})$ and the action extends by linearity to an action on $\mathcal{G}(g,\psi_{\mathbb{C}}\cdot\psi)$. As in §4, we may define a continuous projective representation $\mathbb{P}(\omega_{\psi_{\mathbf{A}}})$ of $G(\mathbf{A})$ on $\mathcal{S}(\mathbb{R}^g)\otimes\hat{\mathcal{S}}^g$ which lifts to a representation $\omega_{\psi_{\mathbf{A}}}$ of an extension $\tilde{G}(\mathbf{A})$ of $G(\mathbf{A})$ by $\mathbb{C}^\times$. The representation $\omega_{\psi_{\mathbf{A}}}$ satisfies

$$\omega_{\psi_{\mathbf{A}}}(\gamma)\rho\psi_{\mathbf{A}}(\sigma)\omega_{\psi_{\mathbf{A}}}(\gamma^{-1}) = \rho\psi_{\mathbf{A}}(\gamma\sigma), \ \gamma \in \tilde{G}(\mathbf{A}), \sigma \in \mathcal{G}(g,\psi_{\mathbf{A}}) \tag{6.4.1}$$

We similarly define the extension $\tilde{G}(\mathbb{R})$ of $G(\mathbb{R})$ by $\mathbb{C}^\times$ to be the pullback of the extension $\tilde{G}(\mathbf{A})$ to the subgroup $G(\mathbb{R})$ of $G(\mathbf{A})$. Then $\tilde{G}(\mathbb{R})$ acts naturally on $\mathcal{S}(\mathbb{R}^g)$, and satisfies the analogue of (6.4.1).

The imbedding $G(\mathbb{Q}) \to G(\mathbf{A})$ lifts to an imbedding $G(\mathbb{Q}) \to \tilde{G}(\mathbf{A})$.

Let $K_h \subset G(\mathbb{R})$ denote the centralizer of $h(\underline{S})$, with h as in 6.0; let $\tilde{K}_h$ be the inverse image of K_h in $\tilde{G}(\mathbb{R})$. Thus $\tilde{G}(\mathbb{R})/\tilde{K}_h \simeq \mathfrak{S}^\pm$; this defines a complex structure on $\tilde{G}(\mathbf{A})/\tilde{K}_h$, and the moduli space $\mathcal{M}_\psi^g$ is naturally isomorphic to $G(\mathbb{Q})\backslash\tilde{G}(\mathbf{A})/\tilde{K}_h$ [,].

Let $\mathfrak{g} = Lie(G(\mathbb{R}))$. The adjoint representation of $GSp(\mathbb{R})$ on $Lie(GSp(\mathbb{R}))$ leaves $\mathfrak{g}$ invariant. Thus $Ad\circ h$ defines a Hodge structure on $\mathfrak{g}$, which is of type $(0,0)+(-1,1)+(1,-1)$ []. Let $\mathfrak{k}_h = \mathfrak{g}^{0,0}$, $\mathfrak{p}^- = \mathfrak{g}^{1,-1}$, $\mathfrak{p}^+ = \mathfrak{g}^{-1,1}$. Then $\mathfrak{k}_h = Lie(K_h)_{\mathbb{C}}$. We let $\tilde{\mathfrak{g}} = Lie(\tilde{G}(\mathbb{R})) = G\oplus\mathbb{C}$, where $\mathbb{C}$ is the center of $\tilde{\mathfrak{g}}$ and $[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}] = \mathfrak{g}$. Let $\tilde{\mathfrak{k}}_h = \mathfrak{k}_h\oplus\mathbb{C}\subset\mathfrak{g}\oplus\mathbb{C}$.

The action of $G(\mathbf{A})$ on $\mathcal{G}(g,\psi_{\mathbb{C}}\cdot\psi)$ pulls back to an action of $\tilde{G}(\mathbf{A})$. Let $\mathbf{D}$ (resp. $\mathbf{D}_{\mathbb{C}}$) be the semidirect product of $\tilde{G}(\mathbf{A})$ and $\mathcal{G}(g,\psi_{\mathbf{A}})$ (resp. $\mathcal{G}(g,\psi_{\mathbb{C}}\cdot\psi)$) with respect to this action, and let $\mathbf{B}_0$ (resp. $\mathbf{B}_{0,\mathbb{C}}$) denote the quotient of $\mathbf{D}$ (resp. $\mathbf{D}_{\mathbb{C}}$) by the subgroup $(t,t)\in Z(\mathcal{G})\times Z(\tilde{G}(\mathbf{A})) = \mathbb{C}^\times\times\mathbb{C}^\times$. Then $\mathbf{B}_0$ acts naturally on $\mathcal{S}(\mathbb{R}^g)\otimes\hat{\mathcal{S}}^g$; we denote this representation $\mathbf{B}_0(\omega)$.

Denote by $d\omega_{\psi_{\mathbb{C}}}$ the natural action of $U(\mathfrak{g}_{\mathbb{C}})$ on $\mathcal{S}(\mathbb{R}^g)$ or on $\mathcal{S}(\mathbb{R}^g) \otimes \hat{\mathcal{S}}^g$. It is known ([],) that

$$d\omega_{\psi_{\mathbb{C}}}(\mathfrak{p}^-)\Phi_h = 0; \tag{6.4.2}$$

(6.4.3) There exists a character $[\frac{1}{2}] : \tilde{K}_h \to \mathbb{C}^\times$ such that

$$\omega_{\psi_{\mathbb{C}}}(k)\Phi_h = [\frac{1}{2}](k)\Phi_h.$$

Let $\Gamma = \underline{H}(\mathbb{Q}) \rtimes G(\mathbb{Q}) \subset \mathbf{B}_0$, $H_{\mathcal{A}} = \tilde{K}_h \subset \mathbf{B}_0$, $H_{\mathcal{A},\mathbb{C}} = Q^- \rtimes \tilde{K}_h \subset \mathbf{B}_{0,\mathbb{C}}$; thus $\hat{\mathcal{A}}^g_\psi(\mathbb{C})$ is analytically isomorphic to $\Gamma\backslash\mathbf{B}_0/H_{\mathcal{A}} \simeq \Gamma\backslash\mathbf{B}_{0,\mathbb{C}}/H_{\mathcal{A},\mathbb{C}}$ [,].

We may apply the theory of 6.1 with $G = \mathbf{B}_0$, $H = H_{\mathcal{A}}$, $\Gamma = \Gamma$, $\mathfrak{q}^+ = \mathfrak{u}_h^+ \oplus \mathfrak{p}^+$, $\mathfrak{q}^- = \mathfrak{u}_h^- \oplus \mathfrak{p}^-$, $M = \hat{\mathcal{A}}^g_\psi(\mathbb{C})$. Extend the character $\rho(-1)$ of 6.2 to $\rho(-1)_{\mathcal{A}} :\ H_{\mathcal{A}} \to \mathbb{C}^\times$:

$$\rho(-1)_{\mathcal{A}} \mid_{Z(\mathcal{G})} = \rho(-1), \rho(-1)_{\mathcal{A}}(k) = [\frac{1}{2}](k)^{-1}, k \in \tilde{K}_h.$$

Following the convention of (6.1.1), we let Θ^{an}_ψ be the bundle $[\rho(-1)_{\mathcal{A}}]$. So far Θ^{an}_ψ is defined only as a C^∞ vector bundle. We define a holomorphic structure on Θ^{an}_ψ by (6.1.3). That this determines a holomorphic structure on Θ^{an}_ψ is verified as in [, p. 40].

It is clear as in [M3] that Θ^{an}_ψ is the analytic vector bundle associated to the algebraic line bundle $\Theta_\psi(\mathcal{A}) = \Theta(\mathcal{A}) \mid_{\hat{\mathcal{A}}^g_\psi}$, with $\Theta(\mathcal{A})$ as in 5.4. Just as in 6.2, there is a homomorphism

$$S_\psi : \mathcal{S}^g \to \Gamma(\hat{\mathcal{A}}^g_\psi, \Theta_\psi(\mathcal{A}))$$

which intertwines the actions of $\mathbf{B}_0(\mathbf{A}_f) \overset{def}{=} \mathcal{G}(g,\psi) \cdot \tilde{G}(\mathbf{A}_f) \subset \mathbf{B}_0$ on both sides. When $g > 1$, S_ψ is an isomorphism by Proposition 2.9 (or by the direct limit version of Proposition 3.11). We exclude the case $g = 1$ from the following discussion; it will be treated in §7.

Let $U_{\mathcal{A}}$ denote the enveloping algebra of $Lie(\mathcal{G}(g,\psi_{\mathbb{C}})) \rtimes G_{\mathbb{C}}$. Let $\mathbb{C}(1)$ denote $\mathbb{C}$ with the action $\rho(-1)^\star$ of $H_{\mathcal{A},\mathbb{C}}$. There is a $\mathbf{B}_0(\mathbf{A}_f)$-equivariant isomorphism of analytic vector bundles (6.1.6):

$$J^\infty(\Theta^{an}_\psi) \simeq \Gamma\backslash\mathbf{B}_0 \times (U_{\mathcal{A}} \otimes_{U(\mathfrak{h}_{\mathcal{A},\mathbb{C}})} \mathbb{C}(1))^\star/H_{\mathcal{A}} \tag{6.4.4}$$

where $\mathfrak{h}_{\mathcal{A},\mathbb{C}} = Lie(H_{\mathcal{A}})_{\mathbb{C}}$ and the dual on the right hand side is taken as an inverse limit.

Denote by $d\Omega_\psi$ the representation of $U_{\mathcal{A}}$ on $\mathcal{S}(\mathbb{R}^g)$ which coincides with $d\rho_{\psi_{\mathbb{C}}}$ (resp. with $d\omega_{\psi_{\mathbb{C}}}$) on $Lie(\mathcal{G}(g,\psi_{\mathbb{C}}))$ (resp. on $U(\mathfrak{g}_{\mathbb{C}})$). The function Φ_h defines a map

$$U_{\mathcal{A}} \to \mathcal{S}(\mathbb{R}^g);\ D \mapsto d\Omega_\psi(D)\Phi_h$$

which factors through $U_{\mathcal{A}} \otimes_{U(H_{\mathcal{A}},\mathbb{C}))} \mathbb{C}(1)$ ((6.2.5), (6.4.2), (6.4.3)). By duality, this map defines a holomorphic subbundle

$$\mathcal{S}^{\infty,an}_\psi \overset{def}{=} \Gamma\backslash\mathbf{B}_0 \times \hat{\mathcal{S}}(\mathbb{R}^g)/H_{\mathcal{A}} \subset J^\infty(\Theta^{an}_\psi).$$

Here the action of $\tilde{K}_h$ on $\hat{\mathcal{S}}(\mathbb{R}^g)$ is the contragredient of $\omega_{psi_{\mathbb{C}}}$

In the notation of 5.6, let $\mathcal{D}$ denote the algebra of differential operators on $\hat{\mathcal{A}}_{\psi^g}$, and let $\mathcal{E}^\infty(\Theta_\psi)$ denote the $\mathcal{O}_{\hat{\mathcal{A}}_{\psi^g}}$-subbundle of $J^\infty(\Theta_\psi)$ generated by $j^\infty(\Gamma(\hat{\mathcal{A}}_{\psi^g}, \Theta_\psi(\mathcal{A})))\star \mathcal{D}$. Let $p : \Gamma\backslash\mathbf{B}_0 \rightarrow \hat{\mathcal{A}}_{\psi^g}(\mathbb{C})$ be the projection. The fiber of $\mathcal{E}^\infty(\Theta_\psi)$ at the point $p(g)$ is canonically isomorphic (6.1.10) to the subspace of $Hom(U_{\mathcal{A}}, \mathbb{C})$ given by

$$\{D \in d\Omega_\psi(X \cdot D)(\Phi_h \otimes \Phi^f)(g),\ D \in U_{\mathcal{A}} \mid\ X \in U_{\mathcal{A}}, \Phi^f \in \mathcal{S}^g\}.$$

It follows that $\mathcal{E}^\infty(\Theta_\psi)^{an} \subset \mathcal{S}_\psi^{\infty,an}$. On the other hand, for any point $x \in \mathcal{M}_\psi^g$, let $\hat{A}_x$ be the fiber of $\hat{\mathcal{A}}_{\psi^g}$ over x, and let $\mathcal{E}_x^\infty$ (resp. J_x^∞) denote the pullback of $\mathcal{E}^\infty(\Theta_\psi)$ (resp. $J^\infty(\Theta_\psi)$) to $\hat{A}_x$. Then Corollary 6.3.2 states exactly that $\mathcal{E}_x^\infty = J_x^\infty$. By dimension considerations, it follows that $\mathcal{E}^\infty(\Theta_\psi) = \mathcal{S}_\psi^{\infty,an}$.

Recall that in 5.6 we defined a bundle $\mathcal{S}^\infty(\Theta)$ over $\tilde{\mathcal{M}}^g$. Let $\mathcal{S}^\infty(\Theta_\psi)$ denote the restriction of $\mathcal{S}^\infty(\Theta)$ to $\mathcal{M}_\psi^g$. Let $\varepsilon : \mathcal{M}_\psi^g \rightarrow \hat{\mathcal{A}}_{\psi^g}$ denote the zero section. The above discussion may be summarized as follows:

Proposition 6.5. . *There is a natural $\tilde{G}(\mathbf{A}_f)$-equivariant isomorphism*

$$(\mathcal{S}^\infty(\Theta_\psi))^{an} \xrightarrow{\sim} \varepsilon * (\mathcal{S}_\psi^{\infty,an}) \stackrel{def}{=} G(\mathbb{Q})\backslash\tilde{G}(\mathbf{A}) \times \hat{\mathcal{S}}(\mathbb{R}^g)/\tilde{K}_h$$

of analytic vector bundles. Here the action of $\tilde{K}_h$ on $\hat{\mathcal{S}}(\mathbb{R}^g)$ is the contragredient of $\omega_{\psi_{\mathbb{C}}}$, and the holomorphic structure on $\mathcal{S}_\psi^{\infty,an}$ is determined as in 6.1 by the action of $\mathfrak{p}^-$ given by the contragredient of $d\omega_{\psi_{\mathbb{C}}}$.

6.5.1. Remark. The point of the above proposition is that it defines a rational structure on the analytic vector bundle $G(\mathbb{Q})\backslash\tilde{G}(\mathbf{A}) \times \hat{\mathcal{S}}(\mathbb{R}^g)/\tilde{K}_h$. The above argument actually proves that the vector bundle $\mathcal{S}_\psi^{\infty,an}$ has a natural rational structure; this may be of some future use.

6.6. It remains to extend the above theory to $\tilde{G}Sp$. Note that the above theory is only defined when ψ is the non-archimedean component of a global additive character. However, for any ψ, we clearly have

$$\tilde{M}_g \simeq \mathcal{M}_\psi^g \times^{\tilde{G}(\mathbf{A}_f)} \tilde{G}Sp(\mathbf{A}_f),\ \mathcal{S}^\infty(\Theta) = \mathcal{S}^\infty(\Theta_\psi) \times^{\tilde{G}(\mathbf{A}_f)} \tilde{G}Sp(\mathbf{A}_f), \tag{6.6.1}$$

where, if $H \subset H'$ is an inclusion of groups and X is a space with right H-action, then $X \times^H H' = X \times H'/H$ is the minimal extension of X to a space with H'-action. Combining (6.6.1) and Proposition 6.5, we obtain a $\tilde{G}Sp(\mathbf{A}_f)$-equivariant isomorphism

$$\mathcal{S}^\infty(\Theta)^{an} \simeq G(\mathbb{Q})\backslash\tilde{G}(\mathbb{R}) \times \tilde{G}Sp(\mathbf{A}_f) \times \hat{\mathcal{S}}(\mathbb{R}^g)/\tilde{K}_h \tag{6.6.2}$$

of analytic vector bundles.

Let $\Gamma = G(\mathbb{Q})\backslash[\tilde{G}(\mathbb{R}) \times \tilde{G}Sp(\mathbf{A}_f)]$. It follows from (6.6.2) that we have an isomorphism (cf. (6.1.2))

(6.6.3)

$$\underline{Lift} : \Gamma^\infty(\tilde{M}_g, \mathcal{S}^\infty(\Theta)) \xrightarrow{\sim} \{f \in C^\infty(\Gamma, \hat{\mathcal{S}}(\mathbb{R}^g)) \mid f(gk) = \omega\psi^(k)^{-1}f(g), g \in \Gamma, k \in \tilde{K}_h\}$$

We would like to realize sections of $\mathcal{S}^\infty(\Theta)$ as functions on $\tilde{G}Sp(\mathbf{A})$, defined as in §5 as the semi-direct product of $\tilde{G}(\mathbf{A})$ with the torus $T(\mathbf{A})$. This can be done in a number of ways, depending on the choice of a character of $\mathbb{R}_+^\times$, as follows: Let $GSp(\mathbb{R})^+ = \nu^{-1}(\mathbb{R}_+^\times) \subset GSp(\mathbb{R})$, where ν is the symplectic multiplier, as in §5. Then $GSp(\mathbb{R})^+ \simeq G(\mathbb{R}) \times Z^+$, where Z^+ is the identity component of $Z_{GSp(\mathbb{R})}$, and is isomorphic to $\mathbb{R}_+^\times$. The covering group $\tilde{G}Sp(\mathbb{R})$ admits a natural surjective map to $GSp(\mathbb{R})$. Let $\tilde{G}^+ \subset \tilde{G}Sp(\mathbb{R})$ denote the inverse image of $GSp(\mathbb{R})^+$ under this map; then

$$\tilde{G}^+ \simeq \tilde{G}(\mathbb{R}) \times Z^+ \simeq \tilde{G}(\mathbb{R}) \times \mathbb{R}_+^\times. \tag{6.6.4}$$

Let $f = \underline{Lift}(s)$ be a function on Γ in the image of $\underline{Lift}$, as in (6.6.3), and let α be a character of $\mathbb{R}_+^\times$. The decomposition (6.6.4) defines an extension of f to a function $f_\alpha = \underline{Lift}_\alpha(s)$ on the subgroup $\Gamma^+ \overset{def}{=} G(\mathbb{Q})\backslash\tilde{G}^+ \times \tilde{G}Sp(\mathbf{A}_f) \subset \tilde{G}Sp(\mathbf{A})$:

$$f_\alpha(\tilde{g}z, g^f) = \alpha(z)f(\tilde{g}, g^f), z \in Z^+, \tilde{g} \in \tilde{G}(\mathbb{R}), g^f \in \tilde{G}Sp(\mathbf{A}_f). \tag{6.6.5}$$

Let $GSp(\mathbb{Q})^+ = GSp(\mathbb{Q}) \cup \tilde{G}^+$. We say $s \in \Gamma^\infty(\tilde{M}_g, \mathcal{S}^\infty(\Theta))$ is α-*admissible* if $\underline{Lift}_\alpha(s)$ satisfies

$$\underline{Lift}\alpha(s)(\gamma g) = \underline{Lift}_\alpha(s)(g), \forall \gamma \in GSp(\mathbb{Q}) +, g \in \Gamma^+. \tag{6.6.6}$$

We may identify

$$GSp(\mathbb{Q})^+\backslash\tilde{G}^+ \times \tilde{G}Sp(\mathbf{A}_f) \simeq GSp(\mathbb{Q})\backslash\tilde{G}Sp(\mathbf{A});$$

thus if s is α-admissible, then $\underline{Lift}_\alpha(s)$ is naturally a function on $GSp(\mathbb{Q})\backslash\tilde{G}Sp(\mathbf{A})$, which we denote $\underline{Lift}_{GSp}(s)$. If s is α-admissible, then α is uniquely determined; thus there is no danger of ambiguity.

More generally, we say s is admissible if is a finite linear combination $\sum s_i$, where each s_i is α_i-admissible for some α_i. We let

$$\underline{Lift}_{GSp}(s) = \sum \underline{Lift}_{GSp}(s_i) \ \in \ C^\infty(GSp(\mathbb{Q})\backslash\tilde{G}Sp(\mathbf{A}), \hat{\mathcal{S}}(\mathbb{R}^g)).$$

The subspace $\Gamma^\infty(\tilde{M}_g, \mathcal{S}^\infty(\Theta))_{adm} \subset \Gamma^\infty(\tilde{M}_g, \mathcal{S}^\infty(\Theta))$ of admissible sections is stable under the action of $\tilde{G}Sp(\mathbf{A}_f)$.

It is known [] that the representation $\omega = Ind_{\tilde{G}(\mathbf{A})}^{\tilde{G}Sp(\mathbf{A})}\omega_{\psi_{\mathbf{A}}}$ is independent of the character $\psi_{\mathbf{A}}$. It is thus clear from (6.6.1) that, for any ψ, the action of $\tilde{G}Sp(\mathbf{A}_f)$ on $\Gamma^\infty(\tilde{M}_g, \mathcal{S}^\infty(\Theta)$ is given by the restriction ω_f of ω to $\tilde{G}Sp(\mathbf{A}_f)$. Let $\omega_{adm,\mathbb{R}}$ be the representation of $\tilde{G}Sp(\mathbb{R})$ on the linear space spanned by

$$\{f \in Ind_{\tilde{G}(\mathbb{R})}^{\tilde{G}Sp(\mathbb{R})}\omega_{\psi_\infty} \ \mid \ \exists\ \alpha \text{ such that } f(\tilde{g}z) = \alpha(z)f(\tilde{g}), z \in Z^+, \tilde{g} \in \tilde{G}(\mathbb{R})\}.$$

Then $\omega_{adm,\mathbb{R}}$ is independent, up to isomorphism, of the choice of ψ_∞. Let ω_{adm} denote the subrepresentation $\omega_{adm,\mathbb{R}} \otimes \omega_f$ of ω. It follows from the preceding discussion that the natural representation of $\tilde{G}Sp(\mathbf{A})$ on $\Gamma^\infty(\tilde{M}_g, \mathcal{S}^\infty(\Theta)_{adm})$ is given by ω_{adm}.

7. Fourier Expansions

For each integer $N \geq 1$, let $S_N = S_{g,N}$ be the split torus over $\mathbb{Q}$ with character group $B_N = Hom(Sym^2(\mathbb{Z}^g), (1/N)\mathbb{Z})$; let $B = B_1$. Then for each ψ, S_N is the torus corresponding by Mumford's theory of toroidal compactification to the standard point boundary component of $\mathcal{M}^g_{N,\psi}$. Specifically, the positive-definite symmetric matrices form a cone $C \subset B \otimes \mathbb{R}$ which is homogeneous with respect to the natural action of $GL(g, \mathbb{R})$. For each integer $N \geq 1$, let $\Gamma_\ell(N) \subset GL(N, \mathbb{Z})$ be the principal congruence subgroup of level N. Ash et al. [AMRT] consider $\Gamma_\ell(N)$-invariant decompositions Σ of C into rational polyhedral cones

$$C = \bigcup_{\sigma \in \Sigma} \sigma$$

satisfying certain axioms of finiteness and compatibility. To each such Σ corresponds a locally finite equivariant torus imbedding $S_N \hookrightarrow S_{N,\Sigma}$ which forms a local model near part of the boundary of $\mathcal{M}^g_{N,\psi}$.

Similar considerations hold for all locally symmetric varieties. However, in the case of $\mathcal{M}^g$, the local parametrization of the boundary admits an algebraic interpretation in terms of degenerating abelian varieties. This interpretation was developed in characteristic zero by Brylinski [B] and over $\mathbb{Z}$ by Faltings and Chai [FC]. The latter two authors make explicit use of the arithmetic theory of theta functions; their results include in passing a treatment of the Fourier expansions of forms of weight $\frac{1}{2}$. Let

$$S_\infty = \varprojlim_N S_N$$

where the surjective morphism $S_N \to S_M$, $M \div N$, corresponds to the inclusion $B_M \subset B_N$ of character groups. Then S_∞ is a pro-algebraic torus with character group $B_\infty = Hom(Sym^2(\mathbb{Z}^g), \mathbb{Q})$. Fix a $GL(g, \mathbb{Z})$-invariant rational polyhedral decomposition Σ of C, satisfying the axioms of [AMRT]. Then, for each N, $\Sigma is \Gamma_\ell(N)$-invariant; thus Σ defines a torus imbedding

$$S_\infty \hookrightarrow S_{\infty,\Sigma} = \varprojlim_N S_{N,\Sigma}$$

Let $\hat{S}_{N,\Sigma}$ denote the formal completion of $S_{N,\Sigma}$ at $D_N \overset{def}{=} S_{N,\Sigma} - S_N$, let $S^0_N = \hat{S}_{N,\Sigma} - D_N$, and let $S_\infty = \varprojlim_N S^0_N$. Mumford's theory of degenerating abelian varieties ([M4], reprinted in [FC]) provides a pair consisting of an abelian scheme $\pi : \mathcal{A}^0 \to S^0_1$ and a line bundle $\mathcal{L}^0$ on $\mathcal{A}^0$ defining a principal polarization on $\mathcal{A}^0$. The pair $(\mathcal{A}^0, \mathcal{L}^0)$ has the following properties:

(7.1) $\pi_* \mathcal{L}^0$ is canonically isomorphic to $\mathcal{O}_{S^0_1}$ (by construction; cf. remarks in [F], pp. 354-355, where $\mathcal{L}^0$ is called $\underline{N}$).

(7.2) For $M \div N$, let $p_{N,M} : S^0_N \to S^0_M$ be the natural projection. In view of (7.1), any section $\hat{\theta} \in \Gamma(S^0_N, (p^0_{N,1}) * (\pi_* \mathcal{L}^0))$ can be written as a formal series, convergent in the topology defined by the sheaf of ideals corresponding to D_N:

$$\hat{\theta} = \sum_{\chi \in B_N} a(\chi)\chi.$$

(Cf. [F], p. 326, [FC,**].

(7.3) The subgroup $p_{N,1}^*(\mathcal{A}^0)[N]$ is canonically isomorphic to $\mu_N^g \times (\mathbb{Z}/N\mathbb{Z})^g$. In particular, given an isomorphism $\psi : \mathbf{A}_f/\hat{\mathbb{Z}} \xrightarrow{\sim} \mu_\infty$, we obtain a compatible family of canonical level N structures

$$\alpha_{N,\psi} : (\mathbb{Z}/N\mathbb{Z})^{2g} \xrightarrow{\sim} p_{N,1}^*(\mathcal{A}^0)[N].$$

(7.4) Let $p_{\infty,1} : S_\infty^0 \to S_1^0$, $\pi_\infty : (p_{\infty,1})^*\mathcal{A}^0 \to S_\infty^0$ be the canonical morphisms. The triple $((p_{\infty,1})^*\mathcal{A}^0, (p_{\infty,1})^*\mathcal{L}^0, \{\alpha_{N,\psi}\})$ defines a morphism

$$j\psi : S_\infty^0 \to \mathcal{M}_\psi^g.$$

Then

$$j_\psi^*(\Theta_\psi) \text{ is canonically isomorphic to } \pi_{\infty,*}(p_{\infty,1})^*\mathcal{L}^0.$$

7.5. Let

$$\mathcal{F}_\Theta = \{ \sum_{\chi \in B_\infty} a(\chi)\chi \ \mid \ \exists\ N > 0 \text{ such that } a(\chi) = 0 \text{ for } \chi \notin \bar{C} \cap B_N\}$$

where $\bar{C}$ is the closure of C in $B \times \mathbb{R}$. The above remarks allow us to define a homomorphism

$$F : \Gamma(\mathcal{M}_\psi^g, \Theta_\psi) \to \mathcal{F}_\Theta,$$

except when $g = 1$, which we consider below. Remember that B_∞ consists of symmetric bilinear forms on $\mathbb{Q}^g$, thus each χ is associated to a symmetric $g \times g$ matrix $M(\chi)$ with rational entries. If we view χ as the function $e^{2\pi i Tr(M(\chi)Z)}$ on the Siegel upper half plane, then F becomes the usual Fourier expansion ([F], §9, [FC]), and in particular is injective.

7.6. When $g = 1$, we can define, for any open subset $U \subset \mathcal{M}_\psi^1$, a homomorphism $F^1(\infty) : \Gamma(U, \Theta_\psi) \to \mathcal{F}_\Theta^1$, where

$$\mathcal{F}_\Theta^1 = \{ \sum_{\chi \in B_\infty} a(\chi)\chi \ \mid \ \exists\ N > 0 \text{ such that } a(\chi) = 0 \text{ for } \chi \notin B_N\}.$$

The homomorphism $F^1(\infty)$ is the Fourier expansion at the cusp ∞; for any other cusp c, there is a similar homomorphism $F^{1,c}$. Let

$$\tilde{\Gamma}(U, \Theta_\psi) = \{f \in \Gamma(U, \Theta_\psi) \ \mid \ F^{1,c}(f) \in \mathcal{F}_\Theta^1, \text{ for all cusps } c\}.$$

The variety $\mathcal{M}_\psi^1$ has a natural $G(\mathbf{A}_f)$-equivariant compactification $j : \mathcal{M}_\psi^1 \to \bar{\mathcal{M}}_\psi^1$, obtained by adding points. Let $\bar{\Theta}_\psi \subset j_*\Theta_\psi$ be the subsheaf such that $\Gamma(V, \bar{\Theta}_\psi) = \tilde{\Gamma}(V \cap \mathcal{M}_\psi^1, \Theta_\psi)$, for an open subset $V \subset \bar{\mathcal{M}}_\psi^1$. The arguments of §4-§6 go through also in the case $g = 1$, provided $\mathcal{M}_\psi^1$ is replaced by $\hat{\mathcal{M}}_\psi^1$ and Θ_ψ by $\bar{\Theta}_\psi$. In the future, these modifications for the case $g = 1$ will be assumed without further mention.

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