REPRESENTATION THEORY W4044

HOMEWORK, WEEK 9, DUE APRIL 8

1. Read the proofs of Proposition 20.7 and Theorem 20.8 of the James-Liebeck book; then do Chapter 20, exercise 3 (not related to these results), Chapter 21, 1, 2.

2. Let G be a finite group with subgroup H of index 2. Let $\alpha : G/H \to \mathbb{C}^{\times}$ be the non-trivial character. The following results are proved in Chapter 20 of the James-Liebeck book, by other methods.

(a) Let (ρ, V) be an irreducible representation of G. Suppose $\rho \otimes \alpha$ and ρ are equivalent. Then $\chi_{\rho}(g) = 0$ if $g \notin H$.

(b) Use characters and Frobenius reciprocity to prove the following fact: if $\rho \otimes \alpha$ and ρ are equivalent, then there is a subrepresentation $(\sigma, W) \subset (res_H^G \rho, V)$ such that $\rho \simeq ind_H^G \sigma$. Moreover, $\dim V = 2 \dim W$, $res_H^G \rho = \sigma \oplus \sigma'$ where σ' and σ are inequivalent, and $\rho \simeq ind_H^G \sigma'$.

(c) Conversely, show that, if $\rho \otimes \alpha$ is not equivalent to ρ, then $res_H^G \rho$ is irreducible.

(d) Let n and m be integers. Suppose the symmetric group S_n has a unique irreducible representation (ρ, V) of degree m. Show that ρ is self-dual and its restriction to the alternating group A_n is reducible.

3. Let $H \subset G$ be a subgroup, (ρ, V) a representation of G, (σ, W) a representation of H. Show that

$$ind_H^G(\sigma \otimes res_H^G \rho) \xrightarrow{\sim} ind_H^G(\sigma) \otimes res_H^G \rho.$$