REPRESENTATION THEORY W4044

1. Homework, week 5, due February 25

1. James and Liebeck book, Chapter 13, exercises 5, 7; Chapter 14, exercises 2, 3.

2. Let S_n be the symmetric group on n letters. Let $\rho : S_n \to GL(n, \mathbb{C})$ be the permutation representation. Denote by V the space of ρ.

(a) For any $\sigma \in S_n$, compute the character $\chi_V(\sigma)$ in terms of the cycle decomposition of σ.

(b) Let $V^{S_n} = \mathbb{C}(1, \ldots, 1)$ be the invariant line in V, and let $W = V/V^{S_n}$ be the complementary representation. Use (a) to compute the character χ_W.

3. Let (ρ, V) be a representation of the group G with character χ such that $\chi(e) = 1$, $\chi(g) = 0$ for $g \neq e$. Show that $|G| = 1$.

4. Let G be a finite group and let K be a field. Define $F[G] = F_K[G]$ to be the set of functions from G to K:

$$F[G] = \{f : G \to K\}.$$

If $a \in K$, let $a_G \in F[G]$ be the function such that $a_G(g) = a$ for all $g \in G$ (the constant function with value a).

We define operations of addition and multiplication on $F[G]$: If $f_1, f_2 \in F[G]$, define $f_1 + f_2$ and $f_1 \cdot f_2$ as follows:

$$(f_1 + f_2)(g) = f_1(g) + f_2(g); \ (f_1 \cdot f_2)(g) = f_1(g) \cdot f_2(g).$$

(a) Show that $F[G]$ is a commutative ring with these operations, with additive identity 0_G and multiplicative identity 1_G. (Verify all the axioms, including the condition that $f_1 + f_2$ and $f_1 \cdot f_2$ both belong to $F[G]$.)

(b) Show that $F[G]$ is a field if and only if G is the group with one element.