REPRESENTATION THEORY W4044

1. Homework, week 3, due February 11

1. James and Liebeck book, chapter 3, exercises 3, 5, 6, 7.

(Exercises 3, 5, and 6 concern representations of the dihedral group. Let $n \geq 2$ be an integer. The *dihedral group* D_{2n} is a group of order $2n$ with two generators a, b such that $a^n = e$, $b^2 = e$ (here e denotes the identity element) and the relation $bab^{-1} = a^{-1}$. This can be seen as the group consisting of rotations of the euclidean plane that fix a regular n-gon X inscribed in the unit circle, together with the reflections in the perpendicular bisectors of each of the n-sides of X.)

2. Let V be the n-dimensional vector space K^n over K, with $K = \mathbb{R}$ or $\mathbb{C}$. Let $M(n, K)$ denote the space of $n \times n$ matrices over V. We want to show that, if $L : End(V) = M(n, K) \to K$ is a linear map such that $L(AB) = L(BA)$ for all $A, B \in End(V)$, then L is a scalar multiple of the trace.

(a) First suppose $n = 2$ and define the following matrices:

$$E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Verify that

$$HE_{12} - E_{12}H = 2E_{12};\ HE_{21} - E_{21}H = -2E_{21};\ E_{12}E_{21} - E_{21}E_{12} = H.$$

Deduce that, if $L : End(V) = M(2, K) \to K$ is a linear map such that $L(AB) = L(BA)$ for all $A, B \in End(V)$, then $L(H) = L(E_{12}) = L(E_{21}) = 0$. Thus $\ker L$ is of dimension 3. Show that this implies that L is a scalar multiple of the trace.

(b) Now imitate the argument for $n = 2$ for general n.