REPRESENTATION THEORY W4044

Homework, week 11, due April 22

1. James and Liebeck book, p. 336, exercise 2.

2. Let $H \subset G$, (σ, W) a representation of W. Here is another characterization of the induced representation $I(\sigma) = ind_H^G \sigma$ that can be used to complete the proof of Mackey's decomposition theorem. Suppose (ρ, V) is a representation of G such that

(i) There is a homomorphism $j : W \to V$ such that, for all $h \in H$ and all $w \in W$,

$$j(\sigma(h)w) = \rho(h)j(v).$$

(ii) If $g_i, i = 1, \ldots, r$ are a set of representatives for G/H, then $V = \oplus_i \rho(g_i)(j(W))$.

Prove that ρ is equivalent to $I(\sigma)$, as follows:

(a) Let $W_i = \rho(g_i)(j(W))$. Let $I = \{1, \ldots, r\}$ and identify I with the set of W_i, $i = 1, \ldots, r$. Show that for any $g \in G$, $\rho(g)$ permutes the W_i. Deduce from this a homomorphism from G to S_r, viewed as the group of permutations of I.

(b) For $g \in G$, let I^g be the subset of I fixed by g. If $i \in I^g$, show that $\rho(g)$ defines an automorphism $\rho_i(g)$ of W_i and that

$$\chi_\rho(g) = \sum_{i \in I^g} Tr \rho_i(g).$$

(c) Show that $i \in I^g$ if and only if $(g_i)^{-1} g g_i \in H$.

(d) Show that if $i \in I^g$, then $Tr \rho_i(g) = \chi_\sigma((g_i)^{-1} g g_i)$.

(e) Conclude that ρ is equivalent to $I(\sigma)$ by comparing the characters of the two representations.

3. Let k be a finite field. Let $SL(2,k)$ be the group of 2×2 invertible matrices with coefficients in k and determinant 1. let $PSL(2,k) = SL(2,k)/\{\pm I_2\}$. Let $\mathbb{P}^1(k)$ be the set of one-dimensional subspaces in k^2 (the two-dimensional vector space over k.

(a) Show that $PSL(2,k)$ acts on $\mathbb{P}^1(k)$ by permutations.

(b) Show that $SL(2, \mathbb{F}_2) = PSL(2, \mathbb{F}_2)$ is isomorphic to S_3. Show that $PSL(2, \mathbb{F}_3)$ is isomorphic to A_4.