REPRESENTATION THEORY W4044

Final, due May 8, 2015

All groups are finite and all representations are finite-dimensional, and with complex coefficients, unless indicated otherwise.

1. (a) Find a group G and an irreducible representation (ρ, V) of G such that the values of χ_ρ are all real but for some $g \in G$, $\chi_\rho(g) \notin \mathbb{Q}$.

(b) Describe a procedure for constructing infinitely many examples like the one in (a).

2. (a) The *biregular representation* $(\rho, \mathbb{C}[G])$ of $G \times G$ on $\mathbb{C}[G]$ is the representation

$$\rho(g_1, g_2)f(g) = f(g_1^{-1}gg_2), f \in \mathbb{C}[G], g_1, g_2, g \in G.$$

Compute the character of ρ.

(b) The *conjugation representation* $(c, \mathbb{C}[G])$ of G on $\mathbb{C}[G]$ is the restriction of the biregular representation to the diagonal $G \subset G \times G$, in other words

$$c(g)f(g') = f(g^{-1}g'g),\ f \in \mathbb{C}[G], g, g' \in G.$$

Compute the character of c.

3. Let (ρ, V) be an irreducible representation of G.

(a) Show that $\dim(V \otimes V^*)^G = 1$.

(b) Let $e_1, \ldots, e_n$ be a basis of V. Use the basis to write down a generator of $(V \otimes V^*)^G$.

(c) Let $v \in V$, $v' \in V^*$ be non-zero vectors, and suppose

$$(\rho \otimes \rho^*)(g)(v \otimes v') = v \otimes v'$$

for all $g \in G$. Show that $\dim V = 1$.

4. Let $G = GL(2, \mathbb{F}_q)$ and let $N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{F}_q\} \subset G$ be the upper-triangular nilpotent group. Let $\psi : N \to \mathbb{C}^\times$ be any non-trivial character. Let $(I(\psi), V) = ind_N^G \psi$.

(a) Show that $\chi_{I(\psi)}(g) = 0$ unless $g \in N$.

(b) Compute $\chi_{I(\psi)}(n)$ for $n \in N$. Compute $\langle \chi_{I(\psi)}, \chi_{I(\psi)} \rangle$.

(c) Using the character tables for G, show that, for any irreducible representation ρ of G, $\langle \chi_{I(\psi)}, \chi_\rho \rangle \leq 1$. Use this to determine the number of irreducible constituents of $I(\psi)$ as a representation of G.

5. A certain group G of order 72 has one irreducible representation of degree 8 and at least one additional irreducible representation of degree > 1. Before starting the exercise, prove that the number of elements of any conjugacy class of G divides the order of G.

(a) Determine the number of conjugacy classes of G. Show that all but at most 2 irreducible representations of G have real characters.

(b) Determine the order of the derived subgroup $H = [G, G] \subset G$. Show that if $h \in H$ then every element of the form ghg^{-1} is also in H. Conclude that H is the union of three conjugacy classes of G and determine the number of elements in each. Determine the number of elements in each of the conjugacy classes of G not contained in H. Give names to the conjugacy classes of G and identify them by the number of elements in each one.

(c) Now assume all irreducible representations of G have *real* characters. Write the rows of the character table corresponding to 1-dimensional representations, and the column of the character table corresponding to the unit element.

(d) Suppose (ρ, V) is a representation of G of degree > 1. Show that for every 1-dimensional representation α of G, $\rho \otimes \alpha$ is equivalent to ρ. Deduce that the character of ρ vanishes outside H.

(e) Complete the character table of G.