MODERN ALGEBRA II W4042

Homework, week 7, due October 29

1. View $\mathbb{C}$ as a vector space over $\mathbb{R}$.

(a) Let $w = a + bi \in \mathbb{C}$, $b \neq 0$. Show that $\{1, w\}$ forms a basis for $\mathbb{C}$ over $\mathbb{R}$.

(b) Let $\alpha = c + di \in \mathbb{C}$ be any element, and let $A(\alpha) : \mathbb{C} \to \mathbb{C}$ be the function that takes $z \in \mathbb{C}$ to $\alpha \cdot z$. Show that $A(\alpha)$ is a linear transformation of $\mathbb{C}$ as a $\mathbb{C}$-vector space and as an $\mathbb{R}$-vector space.

(c) Show that every linear transformation of $\mathbb{C}$ as a $\mathbb{C}$-vector space is of the form $A(\alpha)$ with $\alpha \in \mathbb{C}$ as above. Find a linear transformation of $\mathbb{C}$ as an $\mathbb{R}$-vector space that is not $\mathbb{C}$-linear.

(d) Find the matrix of $A(\alpha)$ in the basis $\{1, w\}$ of part (a). Compute its determinant and show that it does not depend on the choice of w. Show that the determinant of $A(\alpha)$ is positive for any α.

2. Let K and L be fields, with $K \subset L$, and suppose L is an n-dimensional vector space over K for a positive integer n.

(a) Let $GL_K(L)$ denote the set of invertible linear transformations of L as a K-vector space. Show that $GL_K(L)$ is a group.

(b) Let $GL_L(L)$ denote the set of invertible linear transformations of L as an L-vector space. Show that $GL_L(L)$ is a subgroup of $GL_K(L)$. Suppose $GL_L(L) = GL_K(K)$. Show then that $n = 1$.

(c) Let H denote the set of ring homomorphisms $\sigma : L \to L$. Show that every element of H is injective. Let $G \subset H$ denote the set of σ such that $\sigma(x) = x$ for all $x \in K$. Show that every $\sigma \in G$ belongs to $GL_K(L)$ and is an isomorphism of rings.