MODERN ALGEBRA II W4042

Homework, week 6, due October 22

1. (a) Let k be a field, and let $R \subset k[T]$ be the set of polynomials whose first derivative vanishes at 0. Show that R is a subring and an integral domain.

(b) Show that $R = k[T^2, T^3]$ and define an isomorphism between R and the ring $R' = k[X, Y]/(X^3 - Y^2)$.

(c) Show that the elements T^2 and T^3 are irreducible in R and use this to deduce that R is not a UFD. (d) Find a non-principal ideal in R.

2. Let p be an odd prime number. If $f, g \in \mathbb{Z}[X]$, we write $f \equiv g \pmod{p}$ if all the coefficients of $f - g$ are divisible by p.

(a) Show that if $f, g, h \in \mathbb{Z}[X]$ and $f \not\equiv 0 \pmod{p}$, then $fg \equiv fh \pmod{p}$ implies that $g \equiv h \pmod{p}$.

(b) Show that $(X-1)^p \equiv X^p - 1 \pmod{p}$ and use this to deduce that

$$(X-1)^{p-1} \equiv c(X) \pmod{p}$$

where $c(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$.

(c) Show that $c(X)$ does not divide $X - 1$ in the ring $\mathbb{Z}/(p)[X]$. Conclude that the principal ideal (p) is not a prime ideal in the ring $\mathbb{Z}[X]/(c(X))$.

3. Show that the polynomials $3X^{10} + 25X^7 - 250X^4 - 30$ and $X^6 + X^3 + 1$ in $\mathbb{Q}[X]$ are irreducible.

4. Rotman's book, p. 49, exercise 68. (You should read pp. 44-49 and try to complete the expression for the roots of the quartic polynomial, finding explicit expressions for the quantities denoted k, ℓ, m on p. 48.)