MODERN ALGEBRA II W4042

Homework, week 5, due October 15

1. (a) Let k be a finite field with q elements, and let V be a vector space over k of dimension n. How many elements does V contain?

(b) Let k be a finite field with 3 elements. Find a polynomial $f \in k[X]$ such that $k[X]/(f)$ has 9 elements.

2. Let $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5}, a, b \in \mathbb{Z}\}$.

(a) Show that R is a subring of $\mathbb{C}$.

(b) Define a bijection $\sigma : R \to R$ by $\sigma(a+b\sqrt{-5}) = a - b\sqrt{-5}$. Show that σ is a homomorphism of rings. Let $N(r) = r\sigma(r)$; show that $N(r) \in \mathbb{Z}$.

(c) Let $r \in R$, $r \notin \mathbb{Z}$, $p \in \mathbb{Z}$, with p prime. Suppose r divides p in R; i.e. $p = rs$ for some $s \in R$. Show that $N(r)$ divides p. (Hint: Show that if $p = rs$ as above then $N(s) > 1$; also, show that $N(r)N(s) = N(rs)$ for any $r, s \in R$.)

(d) Show that if $r \in R$ but $r \notin \mathbb{Z}$ then $N(r) \geq 5$.

(d) Let $r = 1 + \sqrt{-5}$. Show that $6 = 2 \cdot 3 = r \cdot \sigma(r)$. Show that r divides neither 2 nor 3, and that 2 and 3 are irreducible. Conclude that R is not a UFD.

3. Rotman's book, pp. 37-38, exercises 50, 53, 54; p. 43, exercises 63, 65, 66.