MODERN ALGEBRA II W4042

Homework, week 4, due October 8

Let $A = \mathbb{Z}[X]$, the ring of polynomials with integer coefficients. Let $A' = \mathbb{Q}[X]$, and let $\phi : A \to A'$ be the inclusion of A as a subring of A'. Let $J \subset A$ be a maximal ideal.

1. Show that J contains an irreducible polynomial P of degree > 0.

2. Show that J is not a principal ideal. (Hint: Suppose this is false, and say $J = (P)$. We may assume P to be irreducible of degree $n > 0$. Let $a_n \in \mathbb{Z}$ be the leading coefficent of P, and let p be a prime number that does not divide a_n. Let $J' = (P, p)$. Show that the image of J' in $\mathbb{Z}/p\mathbb{Z}[X]$, under the obvious homomorphism

$$A \to \mathbb{Z}/p\mathbb{Z}[X],$$

does not contain 1. Conclude that $J \subsetneq J' \subsetneq A$.)

3. Let K be a field of characteristic zero, i.e. K is a field that contains $\mathbb{Q}$ as a subring. Let A and A' be as above. Let $f : A \to K$ be a ring homomorphism. Let $\phi : A \to A'$ be the natural inclusion, as above. Show that there exists a unique homomorphism $g : A' \to K$ such that

$$f = g \circ \phi.$$

4. Let $P = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X]$. Say P is *primitive* if the ideal $I(P) \subset \mathbb{Z}$ generated by the coefficients $a_0, a_1, \ldots, a_n$ equals $\mathbb{Z}$. Let $Q \in A', Q \neq 0$. Show that Q can be factored as a product

$$Q = c(Q)Q_0$$

where $c(Q) \in \mathbb{Q}^{\times}$ and $Q_0 \in \mathbb{Z}[X]$ is a primitive polynomial.

5. Let L be a field, $g : A' \to L$ a surjective homomorphism. Show that the kernel of g is a principal ideal (Q) where Q is a polynomial of positive degree. Let $Q = c(Q)Q_0$, where $c(Q) \in \mathbb{Q}$ and $Q_0 \in \mathbb{Z}[X]$ is a primitive polynomial, as in the previous question. Show that $(Q) \cap A$ is the principal ideal generated by Q_0.

6. (Optional bonus question) Conclude that there is no surjective homomorphism of the form $A \to L$ if L is a field of characteristic zero.