MODERN ALGEBRA II W4042

Homework, week 2, due September 24

1. Let R be a ring. An *idempotent* in R is an element e such that $e^2 = e$. For example, both 0 and 1 are idempotents.

(a) Suppose R is commutative and has three distinct idempotents. Show that R is not an integral domain.

(b) Suppose R_1 and R_2 are two integral domains. Find the idempotents in the direct product $R_1 \times R_2$.

(c) In the notation of (b), for each idempotent $e \in R_1 \times R_2$, identify the principal ideal $(e) \subset R_1 \times R_2$

2. Let R be an integral domain with fraction field K. A *multiplicative subset* $S \subset R$ is a subset such that,

- $1 \in S$, $0 \notin S$;
- If $s, s' \in S$ then $ss' \in S$.

The *localization* $S^{-1}R$ is the subset of K consisting of elements $\frac{r}{s}$ with $r \in R$ and $s \in S$. (Alternatively, it is the set of equivalence classes of pairs (r, s), with $r \in R$ and $s \in S$, with (r, s) equivalent to (r', s') if and only if $rs' = r's$).

(a) Show that $S^{-1}R$ is a subring of K.

(b) If S is the set of non-zero elements of R, then $S^{-1}R = K$.

(c) Let $R = \mathbb{Z}$. Let p be a prime number, and let S_1 be the set of powers of p:

$$S = \{1, p, p^2, \dots\}$$

and S_2 be the set of integers not divisible by p. Show that S_1 and S_2 are multiplicative sets. Show that $S_1^{-1}\mathbb{Z} \cap S_2^{-1}\mathbb{Z} = \mathbb{Z}$.

(d) Let $I \subset \mathbb{Z}$ be an ideal and let S be either S_1 or S_2 in part (c). Let $S^{-1}I$ be the ideal of $S^{-1}\mathbb{Z}$ generated by I. When does $S^{-1}I = S^{-1}\mathbb{Z}$?

3. Let R be a ring, and consider the polynomial ring $R[X]$. For any $a \in R$, consider the map of sets $ev_a : R[X] \to R$ defined by

$$ev_a(P) = P(a), P \in R[X].$$

Here if $P = \sum_{i=0}^{n} b_i X^i$, with $b_i \in R$, $P(a) = \sum_{i=0}^{n} b_i a^n$.

(a) Show that ev_a is a homomorphism of rings. Find its kernel.

(b) Now suppose R is an integral domain. Let $P \in R[X]$ be a polynomial of degree d. Suppose $P(a) = 0$ and suppose $P = Q_1 Q_2$ with Q_1 and Q_2 both non-constant polynomials. Show that there is a polynomial Q of degree strictly less than d such that $Q(a) = 0$.

4. Rotman's book, exercises 29, 32, 33, p. 20 and exercise 37, p. 23.