MODERN ALGEBRA II W4042

Homework, week 10, due December 3

1. (a) Let $n = p$ or $2p$ where p is a prime number. Let F be any field of characteristic different from p. Show that the Galois group of the polynomial $X^n - 1$ is cyclic.

(b) Show that the Galois group of the extension of $\mathbb{Q}$ generated by the roots of the polynomial $X^{15} - 1$ is not cyclic, and determine the group.

2. Find the Galois group of the polynomial $X^6 - 1$ over each of the finite fields $\mathbb{F}_q$, where $q = 5, 25, 125$.

3. (a) Find all the irreducible polynomials of degree 2 in $\mathbb{F}_2[X]$.

(b) Find an irreducible polynomial of degree 3 in $\mathbb{F}_2[X]$.

(c) Find a general procedure for writing down irreducible polynomials of degree n in $\mathbb{F}_p[X]$ for any p and any n; you don't have to write down the coefficients.

4. Let $f \in \mathbb{Q}[X]$ be a polynomial of positive degree. We consider f such that the graph of the equation $y = f(x)$ in the Cartesian plane is entirely contained in the region $\{(x, y) \in \mathbb{R}^2 \mid y < 0\}$.

(a) Find an example of such an f of degree 4.

(b) Let K be a splitting field for f as above. Show that the set of embeddings of K in $\mathbb{R}$ is empty.