MODERN ALGEBRA II W4042

1. Homework, week 1, due September 17

1. Define a new addition and multiplication on the integers $\mathbb{Z}$ with the following rules:

$$a \;''+''\; b = a + b - 1, a \;''\times''\; b = a + b - ab.$$

where the operations on the right-hand side are the familiar ones. Show that $\mathbb{Z}$, with these operations, is a commutative ring – with additive and multiplicative identities that you have to determine – and that, if neither a nor b equals the additive identity, then $a \;''\times''\; b$ does not equal the additive identity.

2. Let $M(3, \mathbb{R})$ denote the ring of 3×3 matrices with coefficients in $\mathbb{R}$, under usual matrix addition and multiplication. Which of the following subsets of $M(3, \mathbb{R})$ are subrings? Justify your answer. In each case, roman letters designate arbitrary real numbers.

(a) $\left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \right\}$; (b) $\left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} \right\}$; (c) $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$

3. Let $M(n, \mathbb{R})$ denote the ring of $n \times n$ matrices with coefficients in $\mathbb{R}$, under usual matrix addition and multiplication. Define two new operations on $M(n, \mathbb{R})$:

$$X \circ Y = \frac{1}{2}(XY + YX);\; [X, Y] = XY - YX$$

where XY and YX are usual matrix multiplication.

(a) Show that the operation $X \circ Y$ (*Jordan algebra multliplication*) is commutative but not associative.

(b) Show that the operation $[X, Y]$ (*Lie bracket*) is neither commutative nor associative, but is *anti-commutative*: $[X, Y] = -[Y, X]$.

(c) Show that both $X \circ Y$ and $[X, Y]$ are distributive over addition:

$$X \circ (Y + Z) = X \circ Y + X \circ Z;\; [X, Y + Z] = [X, Y] + [X, Z].$$

Does either operation have a multiplicative identity?

(d) Show that the operations $X \circ Y$ and $[X, Y]$ satisfy the following axioms:

$$(X \circ Y) \circ (X \circ X) = (X \circ (Y \circ (X \circ X));$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \text{ the } \textit{Jacobi identity.}$$

4. Show that the intersection of any collection of subrings of a ring R is a subring. Give an example to show that the union of subrings of a ring R is not necessarily a subring.

5. Exercise 28, p. 20 of Rotman's book.