Intro to modern algebra II

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1. Solution to problem set 1

Problem 1. *It is easy to see that the expressions defining the two operations are symmetric in a and b and thus the operations a commutative. We check associativity*

$$(a" +" b)" +" c = (a + b - 1) + c - 1 = a + (b + c - 1) - 1 = a" +" (b" +" c)$$

$$(a" \times" b)" \times" c = (a+b-ab)" \times" c = a+b-ab+c-(a+b-ab)c = a+(b+c-bc)-a(b+c-bc) = a" \times" (b" \times" c)$$

We check distributivity

$$(a" +" b)" \times" c = (a + b - 1) + c - (a + b - 1)c = a + c - ac + b + c - bc - 1 = (a" \times" c)" +" (b" \times" c)$$

The additive identity

$$a" +" 1 = a + 1 - 1 = a$$

The multiplicative identity

$$a" \times" 0 = a + 0 - 0 = a$$

Additive inverses

$$a" +" (2 - a) = a + (2 - a) - 1 = 1$$

Finally, assume that $a" \times" b = 1$*, in other words*

$$0 = a" \times" b - 1 = a + b - ab - 1 = (1 - a)(b - 1)$$

Since $\mathbb{Z}$ *is an integral domain the above implies* $a = 1$ *or* $b = 1$*. Note that this turns our newly defined ring into and integral domain.*

Problem 2. *We observe that the set in part (b) is not closed under multiplication*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The set in (c) is not closed under addition:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & a+c & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally it is easy to check that the set of matrices in (a) forms a ring - the only nontrivial point is to show closure under multiplication. This, however, is a simple exercise in matrix multiplication.

Problem 3. *Observe that*

$$X \circ Y = \frac{1}{2}(XY + YX) = Y \circ X,$$

while

$$X \circ (Y \circ Z) = \frac{1}{4}(XYZ + YZX + XZY + ZYX)$$

and

$$(X \circ Y) \circ Z = \frac{1}{4}(XYZ + ZXY + YXZ + ZYX)$$

These are different, since matrix multiplication is highly noncommutative.

By definition $[X, Y] = XY - YX = -[Y, X]$

$$[X, [Y, Z]] = XYZ - YZX - XZY + ZYX$$

$$[[X, Y], Z] = XYZ - ZXY - YXZ + ZYX$$

Again the two expressions are different.
Since matrix multiplication satisfies distributivity we have

$$X \circ (Y + Z) = \frac{1}{2}(XY + XZ + YX + ZX) = X \circ Y + X \circ Z.$$

Similarly

$$[X, Y + Z] = XY + XZ - YX - ZX = [X, Y] + [X, Z]$$

In part (d)

$$(X \circ Y) \circ (X \circ X) = \frac{1}{2}(XY + YX) X^2 = X \circ (Y \circ (X \circ X))$$

The Jacobi identity:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] =$$
$$(XYZ - YZX - XZY + ZYX) + (ZXY - XYZ - ZYX + YXZ) + (YZX - ZXY - YXZ + XZY) = 0$$

Problem 4. *Let* R_i *be a set of subrings of* R. *Assume that* $x, y \in \bigcap R_i$. *Then* $x, y \in R_i$, *for all* i. *Thus* $x + y \in R_i$ *and* $xy \in R_i$ *for all* i.

Thus $\bigcap R_i$ *is closed under addition and multiplication.*

An easy example whi the union of two subrings need not be a ring: Let $R = \mathbb{Z}$, $R_1 = 2\mathbb{Z}$ *and* $R_2 = 3\mathbb{Z}$. *Assume that* $R' = R_1 \bigcup R_2$ *is a ring. Since* $2 \in R_1$ *and* $3 \in R_2$, $1 = 3 - 2 \in R'$. *However, obviously* $1 \notin R'$.

Problem 5. *Exercise 28*

Let $\varphi : R[X] \to R$ *be defined by* $f(X) \mapsto f(0)$. *It is obvious that if* $f(X), g(X) \in R[X]$ *have constant terms* a_0 *and* b_0 *the constant terms of* $f(X) + g(X)$ *and* $f(X)g(X)$ *are* $a_0 + b_0$ *and* $a_0 b_0$ *respectively.*

Thus φ *is a ring homomorphism. Note that* $\varphi(f) = f(0) = 0$ *if and only if* $f(X)$ *has no free term. Therefore* $ker(\varphi) = (X)$ *- the ideal in* $R[X]$ *generated by* X.