INTRODUCTION TO HIGHER MATHEMATICS V2000

Practice Final Solutions

1. (a) Prove by induction that, for all $n > 0$,

$$\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}.$$

Solution. Write $A(n) = \sum_{i=1}^{n}(-1)^i i^2$, $B(n) = (-1)^n \frac{n(n+1)}{2}$. Base case: for $n = 1$ one sees by inspection that $A(n) = -1$ and $B(n) = -1$.

Induction step: Suppose it is known for n, we prove it for $n+1$. For all n we have

$$A(n+1) = A(n) + (-1)^{n+1}(n+1)^2;$$

$$B(n+1) = B(n) + (-1)^{n+1}\frac{(n+1)(n+2)}{2} - (-1)^n \frac{n(n+1)}{2}$$

$$= B(n) + (-1)^{n+1}[\frac{(n+1)(n+2)}{2} + \frac{(n+1)(n)}{2}].$$

Assuming $A(n) = B(n)$, it thus suffices to show that

$$(-1)^{n+1}(n+1)^2 = (-1)^{n+1}[\frac{(n+1)(n+2)}{2} + \frac{(n+1)(n)}{2}];$$

in other words, that

$$(n+1)^2 = \frac{(n+1)(n+2) + (n)(n+1)}{2}$$

which is easy.

(b) For any $n \geq 1$ let $X_n = \{x \in \mathbb{N}, 1 \leq x \leq n\}$. We consider X_n as a subset of X_{n+1}.

(i) Define an injective map $j : P(X_n) \hookrightarrow P(X_{n+1})$ and describe its image as a subset of $P(X_{n+1})$

(ii) Define a bijection between $j(P(X_n))$ and $P(X_{n+1}) \setminus j(P(X_n))$.

(iii) Prove by induction that $|P(X_n)| = 2^n$.

Solution. (i) Since $X_n \subseteq X_{n+1}$, any subset of X_n is a subset of X_{n+1}. This defines j. The image of j is the set of subsets $S \subseteq X_{n+1}$ such that $n+1 \notin S$.

(ii) By part (i), $P(X_{n+1}) \setminus j(P(X_n)) = \{T \subseteq X_{n+1} \mid n+1 \in T\}$. We define a bijective map

$$\alpha : j(P(X_n)) \to P(X_{n+1}) \setminus j(P(X_n)) \mid \alpha(S) = S \cup \{n+1\}.$$

This is clearly injective and has inverse

$$\beta : P(X_{n+1}) \setminus j(P(X_n)) \to j(P(X_n)) \mid \beta(T) = T \setminus \{n+1\}.$$

(iii) Base case: For $n = 1$, $P(X_1)$ has two elements: X_1 and $\emptyset$.

Induction step: Suppose we know that $|P(X_n)| = 2^n$. Since j is injective, $|j(P(X_n))| = 2^n$. Now

$$|P(X_{n+1}| = |j(P(X_n))| + |P(X_{n+1}) \setminus j(P(X_n))| = 2|j(P(X_n)|$$

because the two subsets of $P(X_{n+1})|$ are in bijection by part (ii). Thus

$$|P(X_{n+1}| = 2|j(P(X_n)| = 2 \cdot 2^n = 2^{n+1}$$

and this completes the induction step.

(c) Prove by induction that for all $n \geq 4$, $n! > 2^n$.

Solution. Base case: For $n = 4$, $n! = 24$, $2^n = 16$.

Induction step: Suppose $n! > 2^n$. Then

$$(n+1)! = (n+1) \cdot n! > (n+1) \cdot 2^n > 2 \cdot 2^n = 2^{n+1}$$

because $n + 1 > 2$.

(d) (Extra credit) Prove by induction that all natural numbers are interesting.

2. (a) Let $I = (-1, 1) \subset \mathbb{R}$ and let $f : I \to [1, \infty)$ be a continuous function. Prove carefully that

$$\lim_{n\to\infty} f((-1)^n \frac{1}{n}) = f(0).$$

Solution. Set $a_n = f((-1)^n \frac{1}{n})$, for $n \geq 1$, and set $L = f(0)$. Let $\varepsilon > 0$. Because f is continuous at 0, there is $\delta > 0$ such that

$$|x - 0| < \delta \Rightarrow |f(x) - L| = |f(x) - f(0)| < \varepsilon.$$

There exists an integer N such that $N > \frac{1}{\delta}$. For $n \geq N$,

$$|(-1)^n \frac{1}{n} - 0| = \frac{1}{n} < \delta.$$

Thus for $n \geq N$,

$$|f(x) - L| < \varepsilon$$

which proves the claim.

(b) Can there be a continuous function $g : I \to \mathbb{R}$ such that

$$g(\frac{1}{n}) = (-1)^n f(\frac{1}{n})?$$

Explain your answer.

Solution. Suppose there were such a function. Then for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$|g(x) - g(0)| < \varepsilon \ \forall x \in (-\delta, \delta).$$

In particular, if $n > \frac{1}{\delta}$ then

$$|(-1)^n f(\frac{1}{n}) - g(0)| < \varepsilon.$$

Take $\varepsilon = 1$. Now for all n, $f(\frac{1}{n}) \geq 1$ by hypothesis, so if n is even then $(-1)^n f(\frac{1}{n}) \geq 1$ but if n is odd then $(-1)^n f(\frac{1}{n}) < -1$. This means that if $g(0) \geq 0$ then $|(-1)^n f(\frac{1}{n}) - g(0)| \geq 1 = \varepsilon$ for all odd n, which contradicts the above inequality; but if $g(0) < 0$ then $|(-1)^n f(\frac{1}{n}) - g(0)| \geq 1 = \varepsilon$ for all even n. So there can be no such continuous g.

3. (a) Let $\mathcal{D}$ denote the set of Dedekind cuts. Define the half-closed interval $[0, 1)$ and the open interval $(0, 1)$ explicitly as subsets of $\mathcal{D}$.

Solution. First, $(0, 1)$ is the set of $L \in \mathcal{D}$ satisfying the following three conditions: first

$$\forall a \in \mathbb{Q} \ (a \leq 0) \Rightarrow a \in L;$$

second, there exists $r \in \mathbb{Q}$, $r < 1$ such that $r \notin L$; and finally, there exists $a \in \mathbb{Q}$, $a > 0$ such that $a \in L$.

Next, $[0, 1)$ is the set of $L \in \mathcal{D}$ that satisfy the first two conditions above but not the last.

(b) (This is a challenging problem, much more difficult than anything you are likely to see on the exam.) Let $a < b$ be rational numbers, and let $f : (a, b) \to \mathbb{R}$ be a continuous function. We suppose there are $a', b' \in (a, b)$, $a' < b'$, such that $f(a') < 0$ and $f(b') > 0$. Finally, we suppose f is *strictly increasing*: if $x, y \in (a, b)$, $x < y \Rightarrow f(x) < f(y)$.

(i) Let $L \subset \mathbb{Q}$ be the set of all rational numbers in $(\infty, a]$, together with the set of all rational numbers $x \in (a, b)$ such that $f(x) < 0$. Show that L is a Dedekind cut. (Hint: suppose L has a maximum, say x_0, and consider $\varepsilon = -\frac{(f(x_0)}{2}$.)

(ii) Show that L, viewed as a real number, belongs to (a, b). Show that $f(L) = 0$.

(To be discussed during the review.)

(c) Extra practice: Work out the exercises 8.10, 8.11, 8.12, 8.13, 8.14, 8.15 in Dumas-McCarthy.

4. (a) Define surjective functions $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{N}$.

Solution. We can take $f(0) = 0$ and for $i > 0$ we take $f(2i-1) = i$, $f(2i) = -i$. We can take $g(x) = x$ for $x > 0$ and $g(y) = 0$ for $y \leq 0$.

(b) Let A, B, and C be the intervals in $\mathbb{R}$ given by $A = (0,1]$, $B = [1,\infty)$, $C = [1,2)$. (i) Construct bijections $f : A \to B$ and $g : A \to C$.

(ii) Construct a collection of sets $C_i, i \geq 1$, and bijections

$$g_i : C \to C_i,\ \ h : B \to \ \cup_{i\geq 1} C_i$$

(iii) Show that A has a bijection with a countable infinite union of copies of itself.

Solution. (i) Define $f(x) = \frac{1}{x}$, $g(x) = 2 - x$.

(ii) Define $C_i = [i, 1+1)$. Then $B = \cup C_i$ is a partition of B so the function h just takes x to itself. Meanwhile, $g_i(x) = x + i - 1$.

(iii) Since A is in bijection with C, and C is in bijection with each C_i, we see that for each i there is a bijection of A with C_i given by $g_i \circ g$. On the other hand, f is a bijection of A with B which is in bijection, via h, with the infinite union of half-open intervals C_i, each of which is a copy of A.

5. (a) (i) Show that for all $x \in \mathbb{R}$, $|x^2 - 1| \leq x^2 + 1$.

(ii) Define $f : \mathbb{R} \to [-1,1)$ by $f(x) = \frac{x^2-1}{x^2+1}$. Show that f is continuous and surjective.

Solution. (i) Either $|x| \leq 1$ or $|x| > 1$. If $|x| \leq 1$ then $0 \leq x^2 \leq 1$ and so $0 \leq |x^2 - 1| \leq 1$, but $x^2 + 1 \geq 1$. On the other hand, if $|x| > 1$ then $|x^2 - 1| = x^2 - 1 < x^2 + 1$.

(ii) It follows from (i) that $|\frac{x^2-1}{x^2+1}| = \frac{|x^2-1|}{x^2+1} \leq 1$. Moreover, $f(x) = 1$ is impossible, so the image of f is contained in the half-open interval $[-1,1)$. It is continuous because it is the quotient of two continuous functions and the denominator never takes the value 0.

To show that f is surjective, we let $a \in [-1,1)$ and solve the equation

$$f(x) = \frac{x^2-1}{x^2+1} = a.$$

This gives us

$$x^2 - 1 = ax^2 + a;\ (1-a)x^2 = a + 1$$

and thus $x = \pm\sqrt{\frac{a+1}{1-a}}$ is a solution, *provided the expression on the right has a square root.* First of all, note that the denominator never equals 0 because $1 \notin [-1,1)$. Now the numerator is never negative if $a \geq -1$, so there is a real square root provided the denominator is positive; and this is true provided $a < 1$. Note that it is impossible for both numerator and denominator to be negative.

(b) (i) Does there exist an injective function $f : \mathbb{N} \to \mathbb{N}$ such that $(\forall i \in \mathbb{N}) f(i) > f(i+1)$? Explain.

(ii) Let I denote the open interval $(0,1) \subset \mathbb{R}$. Does there exist an injective function $f : \mathbb{N} \to I$ such that $(\forall i \in \mathbb{N}) f(i) > f(i+1)$? Explain.

Solution. (i) There is no such function. Suppose there were such an f. Consider $N = f(0)$. It follows by induction that for all $i > 0$, $f(i) < f(0) = N$. Thus f is an injective function from the infinite set $\mathbb{N}\backslash$ to the finite set $\{0, 1, \dots, N\}$, which is impossible.

(ii) Yes. For example, we can take $f(n) = \frac{1}{2^{n+1}}$.

(c) Let Y be the set $\{1, 2, \dots, m\}$. Prove by induction that the cardinality of the set of functions from $\{1, \dots, n\}$ to Y has cardinality m^n.

This has been done in class.

6. (a) Definitions: Define *bound variable*, *free variable*, *tautology*, *characteristic set*.

(b) Use truth tables to show that the following statement is not a tautology.

$$(P \wedge (\neg Q \vee R)) \Leftrightarrow \neg(R \Rightarrow (Q \wedge \neg P)))$$

(One example suffices.)

Solution. If you substitute T for P and F for Q and R you get F.

7. (a) Let $n > 0$ be a positive integer and let $\mathbb{Z}/n\mathbb{Z}$ be the set of congruence classes modulo n. Define a relation R on $\mathbb{Z}/n\mathbb{Z}$:

$$aRb \Leftrightarrow ab \equiv 0 \pmod{n}.$$

Is this relation reflexive, symmetric, or transitive for all n? For some n?

Solution. It is not reflexive for $n > 1$ (it is not true that $1R1$ if $n > 1$) but it is reflexive for $n = 1$. It is symmetric. Since it is not reflexive but $aR0$ for all a it is also not transitive, again except when $n = 1$.

(b) Find $c =$ gcd$(3075, 3649)$ and find $m, n \in \mathbb{Z}$ such that $3075m + 3649n = c$.

Solution. The GCD is 41, computed as follows:

$$\begin{aligned}
3649 - (3075 \times 1) &= 574\\
3075 - (574 \times 5) &= 205\\
574 - (205 \times 2) &= 164\\
205 - (164 \times 1) &= 41\\
164 - (41 \times 4) &= 0
\end{aligned}$$

We work backwards from the next to last line:

$$41 = 205 - 1 \cdot 164 = 205 - 1 \cdot [574 - 2 \cdot 205] = -1 \cdot 574 + 3 \cdot 205.$$

Continuing:

$$41 = -1 \cdot 574 + 3 \cdot [3075 - 5 \cdot 574] = 3 \cdot 3075 - 16 \cdot 574;$$

$$41 = 3 \cdot 3075 - 16 \cdot [3649 - 1 \cdot 3075] = 19 \cdot 3075 - 16 \cdot 3649.$$

8. True or false? Justify your answer.

(a) For (i) $S = \mathbb{R}$ and (ii) $S = \mathbb{N}$, determine the truth or falsity of the following sentence:

$$(\forall x \in S)(\forall y \in S) x < y \Rightarrow (\exists z \in S)(x < z) \wedge (z < y).$$

(i) True (take $z = \frac{x+y}{2}$); (ii) False: if $y = x+1$ there is no natural number in between.

(b) If $a, b \in \mathbb{N}$ then

$$(\forall a \in \mathbb{N})(\forall b \in \mathbb{N})(\forall c \in \mathbb{N})\ c|\gcd(a,b) \Rightarrow (\forall m \in \mathbb{N})\ c|(ma - b).$$

True.

(c) Let $f : (0,2) \to \mathbb{R}$ be a continuous function, where $(0,2)$ is the open interval. Then there is a real number $C > 0$ and $\delta > 0$ such that, for all $x \in (1-\delta, 1+\delta)$, $|f(x)| < C$.

True. Let $M = f(1)$. Let $\varepsilon > 0$ and let $\delta > 0$ be such that

$$|x - 1| < \delta \Rightarrow |f(x) - M| < \varepsilon$$

and take $C = M + \varepsilon$.

(d) Any compound statement is propositionally equivalent to one that contains only atomic statements and the propositional connectives $\neg$ and $\vee$.

True: one can use De Morgan's laws to replace occurrences of $\wedge$ by occurrences of $\neg$ and $\vee$: $P \wedge Q$ is equivalent to $\neg(\neg(P \vee Q))$. Similarly, $P \Rightarrow Q$ is equivalent to $Q \vee \neg P$.