INTRODUCTION TO HIGHER MATHEMATICS V2000

Review for second midterm: solutions

Problems are in blue, solutions in black.

1. (a) State the principle of Strong Induction, defining all terms.

(b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point a.

(c) State the division algorithm, defining all terms.

Solutions: omitted

2. (a) Find a formula for the sum of the first n even integers.

$$E(n) = 2 + 4 + \cdots + 2n.$$

Find it first by using the formula $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Then give a separate proof using mathematical induction.

The first solution:

$$E(n) = 2 + 4 + \cdots + 2n = 2(1 + 2 + \cdots + n) = 2\frac{n(n+1)}{2} = n(n+1).$$

The second solution: Since we already know that the answer is $n(n+1)$, we prove this is true by induction. The case $E(1) = 2 = 1(1+1)$ is true. Suppose we know $E(n) = n(n+1)$. Then

$$E(n+1) = E(n)+2(n+1) = n(n+1)+2(n+1) = (n+1)(n+2) = (n+1)(n+1+1)$$

which completes the induction step.

(b) Find a formula for the sum of the first n odd integers

$$O(n) = 1 + 3 + \cdots + 2n - 1$$

using the results of (a), then prove it using mathematical induction.

The first solution: Define $S(n) = 1+2+\cdots+n$. We know $S(n) = \frac{n(n+1)}{2}$.

$$O(n)+E(n) = 1+2+3+4+\cdots+(2n+1)+2n = S(2n) = \frac{2n(2n+1)}{2} = n(2n+1).$$

Thus

$$O(n) = S(2n) - E(n) = n(2n+1) - n(n+1) = n(2n+1-(n+1)) = n^2.$$

The second solution: Since we already know that the answer is n^2, we prove this is true by induction. The case $O(1) = 1 = 1^2$ is true. Suppose we know $O(n) = n^2$. Then

$$O(n+1) = O(n) + [2(n+1) - 1] = n^2 + (2n+1) = (n+1)^2$$

which completes the induction step.

(c) Exercises 4.23 and 4.24 in Dumas-McCarthy.

Exercise 4.23: We use strong induction on the number of symbols in the statement. For this we need to recall that the symbols are all of the form P_i, Q_k where P_i are atomic statements, and the four connectives $\neg, \wedge, \vee, \Rightarrow$. Let $T(n)$ be the claim that all well-formed propositional statement of at most n symbols has a well-defined truth value. If $n = 0$ or $n = 1$ this is true by definition because an atomic statement by definition is a statement, hence has a truth value. Now suppose $T(m)$ is true for $m < n$. If S is a well-formed statement with n symbols, then by definition it is of the form $\neg P$, $P \wedge Q$, $P \vee Q$, or $P \Rightarrow Q$ where P and Q are well-formed statements. We see that P and Q have fewer than n symbols, hence have well-defined truth values. Now we apply the truth tables to obtain the truth values of the compounds.

Exercise 4.24: This uses De Morgan's laws to replace $\wedge$ by $\vee$ and $\neg$, and the equivalence of $P \Rightarrow Q$ and $Q \vee \neg P$ to eliminate the $\Rightarrow$'s.

3. (a) Let p be a prime number and let b be an integer that is relatively prime to p. For any integer $a \in \mathbb{Z}$, we denote the congruence class of a modulo p by $[a]$. Show that the operation

$$f_b : [a] \mapsto [ba]$$

is a well-defined bijection $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

Done in class.

(b) In the situation of (a), show that there exists $c \in \mathbb{Z}$ such that $bc \equiv 1 \pmod{a}$. Show that the operations f_c and f_b are inverse bijections.

Since f_b is a bijection, there exists c such that $f_b(c) = [1]$, and this exactly means $bc \equiv 1 \pmod{a}$. Now for any a,

$$f_c \circ f_b([a]) = f_c([ba]) = [cba] = [bca] = [1 \cdot a] = [a].$$

Since multiplication mod p is commutative, we similarly obtain $f_b \circ f_c([a]) = [a]$.

(c) Using Fermat's little theorem, show that $c \equiv b^{p-2} \pmod{p}$. NOTE MISPRINT!

We know that $b^{p-1} \equiv 1 \pmod{a}$ by Fermat's little theorem. Thus

$$b^{p-2} \cdot b \equiv 1 \equiv c \cdot b \pmod{p}.$$

It follows that

$$p \mid [c \cdot b - b^{p-2} \cdot b] \Rightarrow p \mid b(c - b^{p-2}).$$

By Gauss's lemma, either $p \mid b$ or $p \mid c - b^{p-2}$. But since b is relatively prime to p, we see that the latter must be true, hence

$$b^{p-2} \equiv c \pmod{p}.$$

(d) Use the Euclidean algorithm to find the greatest common divisor of 247 and 456.

Solution omitted.

4. (a) Let $a_n = \frac{n-1}{2n-1}$. Using the definitions, show that

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$

Solution omitted (material not covered).

(b) Let f be the function

$$f(x) = \frac{x^3}{x-2}$$

on the set $\mathbb{R} \setminus 2$. Show using the definitions that f has no limit at $x = 2$.

Solution: Proof by contradiction. Suppose $\lim_{x\to 2} f(x) = L$ for some $L \in \mathbb{R}$. Then for $\varepsilon = 1$ there is a $\delta > 0$ such that

$$0 < |x-2| < \delta \;\Rightarrow |f(x) - L| < 1.$$

(Since we know that the function is not bounded near $x = 2$, we can take any positive number for ε; the value 1 is the most convenient.)

Now let $a = min(\frac{8}{|L+2|}, \frac{\delta}{2})$ and take $x = 2+a$ (or any number in $(2, 2+a] \subsetneq (2, 2+\delta)$. Since the numerator x^3 is an increasing function, we know that

$$f(x) = \frac{2+a}{a} > \frac{2^3}{a} = \frac{8}{a} \geq |L+2|.$$

In particular, by the triangle inequality

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L|$$

which implies

$$|f(x) - L| \geq |f(x)| - |L| \geq |L+2| - |L| \geq 2 > \varepsilon.$$

This contradicts the hypothesis that L is the limit.

(c) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. Let $a \in \mathbb{R}$ and suppose f and g are continuous at a. Prove that the product $f \cdot g$ is continuous at a.

Solution omitted (proof done in class and also in Dumas-McCarthy.)