MATH V2000, Spring 16. Review for Midterm II.

The exam covers Ch 3.5, Ch 4.1-4.2, Ch 7.1-7.3, Ch 5.1-2 and parts of 5.3 (up to the end of Example 5.22 on sequences). Also we did rather little on continuity and I will NOT ask you explicitly about results from 5.2 (but you are expected to be able to prove the limits calculated in Example 5.15.) There will be four questions, each worth 15 points. The questions below are longer, but I did indicate point values which will roughly correspond to those in the exam itself.

As in Midterm I, I will ask you to state some definitions (eg of well ordered set, limit of sequence and of a function) and also to state some of the basic results. NOTE: if I ask you to prove something from the definition, then that is all you can use. On the other hand, if I ask you to explain an answer, then you can use any argument and quote results from the book etc. but do state the results you use carefully.

Ex 1. (i) (3 points) Let $L, a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a function. Define what it means to say $\lim_{x\to a} f(x) = L$.
(ii) (6 points) Use the definition to prove that $\lim_{x\to -1} 3x^2 - 4 = -1$.
(iii) (6 points) Use the definition to show that $\lim_{x\to 0} 3x + 1 \neq 0$.
(iv) (6 points) Show that limits are unique: i.e. if $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} f(x) = N$ then $M = N$.

Ex 2. (i) (3 points) Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$, and let $L \in \mathbb{R}$. What does it mean to say that $\lim_{n\to\infty} x_n = L$?
(ii) (6 points) Suppose that $(a_n), (b_n)$ are sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Prove that $\lim_{n\to\infty} (a_n + b_n) = L + M$.
(iii) (6 points) Use the definition to prove that $\lim_{n\to\infty} \frac{n+2}{n-1} = 1$.

Ex. 3. (i) (3 points) State the Principle of Mathematical Induction. (This is Cor 4.4)
(ii) (10 points) Use it to prove that $1 + 5 + \cdots + (4k + 1) = (k + 1)(2k + 1)$ for all $k \in \mathbb{N}$.
(iii) (10 points) Consider the Fibonacci sequence $F_1 = 1, F_2 = 1, F - 3 = 2$ and generally $F_{n+1} = F_n + F_{n-1}$. Prove by induction that for all $k \geq 1$ the number F_{5k} is divisible by 5.
Hint: Find a relation between F_{n+5} and F_n.
(iv) (10 points) Prove by induction on n that when $x > 0$

$$(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2 \text{ for all positive integers } n.$$

Ex. 4. (i) (8 points) Let $m, n \in \mathbb{N}$. Show that if a prime p divides the product mn then it divides at least one of m, n. You may use the fact that if $\gcd(a, b) = 1$ then there are integers k, ℓ such that $ka + \ell b = 1$.
(ii) (10 points) Find integers m, n such that $14m + 13n = 7$.

(iii) (10 points) Let $a, b \in \mathbb{N}$. We defined $\gcd(a, b)$ to be the largest integer that divides both a, b. Show that if s divides both a, b then s also divides $\gcd(a, b)$. What is the simplest argument you can find to prove this?

Ex. 5. (5 points each) True or false? Explain your answers as clearly as possible.

(i) Consider the order relation on the set $\mathbb{N} \times \mathbb{N}$ given by $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. This is a well ordering.

(ii) There is a function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x \to 0} f(x)$ exists but $f(\frac{1}{n}) = n$ for all $n \geq 1$.

(iii) There are divergent sequences $(a_n), (b_n)$ such that the sum $(a_n + b_n)$ is convergent. (Here divergent means "has no finite limit", while convergent means "has a finite limit".)

Good exercises: from DMcC: Ex 5.3 (do this by induction); Ex 5.22; Ex 7.1, Ex 4.17, Ex 4.28.

From Daepp-Gorkin: Problem 18.24 (a), Problem 20.8, Problem 20.16, Problem 27.10, Problem 27.18, Probem 27.21, Problem 28.1 (a)