INTRODUCTION TO HIGHER MATHEMATICS V2000

Review for Midterm II, spring 2016: solutions

Problems are in blue, solutions in black.

Ex. 1. (i) - (iii)

Solutions: omitted (definition or done in class).

Ex. 1 (iv): Show that limits are unique.

Solution: Suppose $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} f(x) = N$. We need to show that $M = N$. Instead we will show that, for any $\varepsilon > 0$, $|M - N| < \varepsilon$. Thus the difference between them is smaller than every positive number, and this implies that they must be equal.

By hypothesis, for any $\varepsilon > 0$, there exists δ_1 such that

$$0 < |x - a| < \delta_1 \;\Rightarrow\; |f(x) - M| < \varepsilon/2$$

and

$$0 < |x - a| < \delta_2 \;\Rightarrow\; |f(x) - N| < \varepsilon/2$$

Let $\delta = min(\delta_1, \delta_2)$. Thus

$$0 < |x - a| < \delta \;\Rightarrow\; |f(x) - M| < \varepsilon/2 \text{ AND } |f(x) - N| < \varepsilon/2.$$

It follows that

$$0 < |x - a| < \delta \;\Rightarrow\; |f(x) - M| + |f(x) - N| < \varepsilon.$$

Thus by the triangle inequality

$$|M - N| = |M - f(x) + f(x) - N| \le |f(x) - M| + |f(x) - N| < \varepsilon,$$

which is what we wanted to prove.

Ex. 2.

Omitted because we didn't cover limits of sequences.

Ex. 3

Solutions: We omit 3. (i) which is a definition.

(ii) We prove

$$S(k) = 1 + 5 + \cdots + (4k + 1) = (k + 1)(2k + 1)$$

The case $k = 0$ is obvious. Suppose we know it for k. Then

$$S(k+1) = S(k)+4(k+1)+1 = (k+1)(2k+1)+4(k+1)+1 = (k+2)(2k+3)$$

as one verifies by simple algebra.

(iii) Define the Fibonacci sequence by $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_{n+1} = F_n + F_{n-1}$. Prove by induction that for all $k \ge 1$, F_{5k} is divisible by 5.

Proof by induction: First $F_4 = 2 + 1 = 3$, $F_5 = 3 + 2 = 5$ so it's true for $k = 1$.

Now for general n we know We know that

$$F_{n+5} = F_{n+4} + F_{n+3} = F_{n+3} + F_{n+2} + F_{n+3}.$$

Substituting $F_{n+3} = F_{n+2} + F_{n+1}$ we find

$$F_{n+5} = 3F_{n+2} + 2F_{n+1}.$$

Substituting $F_{n+2} = F_n + F_{n+1}$ we find

$$F_{n+5} = 3(F_n + F_{n+1}) + 2F_{n+1} \equiv 3F_n \pmod 5.$$

Now suppose F_{5k} is divisible by 5; then

$$F_{5(k+1)} = F_{5k+5} \equiv 3F_{5k} \pmod 5$$

is also divisible by 5.

Prove by induction on n that when $x > 0$ we have

$$(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2}x^2.$$

Proof: It's clearly true for $n = 1$. Suppose it's true for n. Then

$$(1 + x)^{n+1} = (1 + x)(1 + x)^n \geq (1 + x)[1 + nx + \frac{n(n-1)}{2}x^2]$$

and when we work out the right-hand side we find this is

$$1+(n+1)x+\frac{n(n-1)}{2}x^2+nx^2+\frac{n(n-1)}{2}x^3 \geq 1+(n+1)x+[\frac{n(n-1)}{2}+n]x^2$$

and it is now obvious that it's true for $n + 1$.

Ex. 4 (i) Omitted

(ii) Find integers m, n such that $14m + 13n = 7$.

Solution: Obviously $14 \cdot 1 + 13 \cdot (-1) = 1$. Multiply both sides by 7 to find the solution $m = 7$, $n = -7$.

(iii) Find the simplest proof of the fact that if we define $gcd(a, b)$ to be the largest integer that divides both a and b, then if $s \mid a$ and $s \mid b$ then s divides the gcd of a and b.

This is a somewhat ambiguous question: what is the "simplest" proof? Probably the proof uses the fact that if we let c be the largest integer dividing both a and b, then there are integers m and n such that

$$c = ma + nb.$$

It's clear that if s divides a and b then s divides $ma + nb$, and therefore divides c. The problem with the wording is that it's not clear whether or not the proof includes the proof that c can be written in the indicated way. If we write $a = ic$ and $b = jc$ then one can prove that i and j have no common factor – otherwise a and b would have a common divisor larger than c (this fact also requires proof, but it is easy). Then Bezout's lemma implies that there are m and n such that $mi + nj = 1$. An easy argument (that nevertheless needs to be written down) then implies $ma + nb = c$.

I'm not sure whether or not this was the expected answer.

Ex. 5. (i) False: This is not even a linear ordering. There is no order relation between $(1, 2)$ and $(2, 1)$.

(ii) False: If $L = \lim_{x \to 0} f(x)$ exists then for $n > \frac{1}{\delta}$ we must have $|f(\frac{1}{n}) - L| < \varepsilon$ when δ and ε are given by the usual conditions. In particular, $f(\frac{1}{n}) \leq L + 1$ for sufficiently large n, but this is contradicted by the assumption.

(iii) Not covered in class. (But the claim is true.)