INTRODUCTION TO HIGHER MATHEMATICS V2000

Review for second midterm

1. (a) State the principle of Strong Induction, defining all terms.

(b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point a.

(c) State the division algorithm, defining all terms.

2. (a) Find a formula for the sum of the first n even integers.

$$E(n) = 2 + 4 + \cdots + 2n.$$

Find it first by using the formula $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Then give a separate proof using mathematical induction.

(b) Find a formula for the sum of the first n odd integers

$$O(n) = 1 + 3 + \cdots + 2n - 1$$

using the results of (a), then prove it using mathematical induction.

(c) Exercises 4.23 and 4.24 in Dumas-McCarthy.

3. (a) Let p be a prime number and let b be an integer that is relatively prime to p. For any integer $a \in \mathbb{Z}$, we denote the congruence class of a modulo p by $[a]$. Show that the operation

$$f_b : [a] \mapsto [ba]$$

is a well-defined bijection $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

(b) In the situation of (a), show that there exists $c \in \mathbb{Z}$ such that $bc \equiv 1 \pmod{a}$. Show that the operations f_c and f_b are inverse bijections.

(c) Using Fermat's little theorem, show that $c \equiv b^{p-2} \pmod{a}$.

(d) Use the Euclidean algorithm to find the greatest common divisor of 247 and 456.

4. (a) Let $a_n = \frac{n-1}{2n-1}$. Using the definitions, show that

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$

(b) Let f be the function

$$f(x) = \frac{x^3}{x-2}$$

on the set $\mathbb{R} \setminus 2$. Show using the definitions that f has no limit at $x = 2$.

(c) Suppose f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. Let $a \in \mathbb{R}$ and suppose f and g are continuous at a. Prove that the product $f \cdot g$ is continuous at a.