V2000: Review for Midterm 1, Feb 18, 2016

This exam will have 4 questions each worth 15 points. All answers should be clearly written with all statements justified (but not necessarily formally proved – you have to use your judgement about that.) NO CALCULATORS, cell phones turned off.

Syllabus: everything in Dumas-McCarthy up to and including Sec 3.4, except for Secs 1.7 and 2.4.

I will ask you to state some definitions and/or to prove some theorems. You need to know all the definitions. Proofs you should know: Proposition 2.18; Proposition 2.23, and the model proofs on the handouts. I may ask you to reproduce parts of these.

There are many other proofs you can take as models: eg the following examples contain easy proofs Examples 1.25, 1.27, 2.5, 2.27, 2.30, 2.31, 3.1.

You are expected to understand Thm 2.20 (i), but I will not ask you about part (ii).

Here are some good exercises that I did not assign for HW:
Ex 1.3, 1.11, 1.22, 1.23, 1.26, 1.31, 1.34, 2.1, 2.2, 2.3, 2.5, 2.10, 2.18, 2.19, 2.22, 3.6, 3.7, 3.16, 3.17 Do not worry about exercise numbered 3.20 and above.

Note: Ex1.27 has a typo: Its last line should have Y not y.

As I said in the email: look at Daepp and Gorkin, particularly Chapters 4, 8, 10,11, 17

Sample questions

Here are some typical questions (except that some are a little short and some are definitely too long...).

Ex 1. (i) Denote by $[a]_n$ the set of all integers congruent to a modulo n. Show that the operation

$$[a]_n + [b]_n = [a+b]_n$$

is well defined.
(ii) Find the last digit of 3^{7^8}.

Ex 2. Consider the statement:

$$\Big(\forall x \in \mathbb{R}\Big), \quad \big(x < 0\big) \implies \big(\exists n \in \mathbb{Z}, x + n > 0\big)$$

(i) Write down its contrapositive, its converse and its negation in as simplified a form as you can.
(ii) Of these four statements, which are true, which are false? Justify your answer.

Ex 3. (i) Define what is meant by saying that a relation R on the set X is

a) transitive, b) symmetric, c) antisymmetric

(ii) Let X be the set of all functions $f : [0, 1] \to \mathbb{R}$, and let R be the relation on X defined by

$$fRg \Longrightarrow f(x) = g(x) \quad \text{for some } x \in [0, 1].$$

d) Sketch the graphs of functions $f, g, h \in X$ such that fRg but $f \not R h$.

e) Which of the properties (a), (b), (c) does this relation have?

f) Given f describe the set $S[f]$ of all functions g such that fRg.

h) If $S[f] = S[g]$ what can you say about f and g?

Ex 4. (i) Let $f : X \to X$ be a function. What does it mean to say that f is injective, surjective?
(ii) Show that if f is surjective so is its composite with itself $f \circ f : X \to X$.
(iii) Show that if f is injective, then for any subsets $A, B \subset X$ we have $f(A \cap B) = f(A) \cap f(B)$.

Ex 5. Let $X = \mathbb{N}^+$. Let us say xRy if $x < y + 2$ and xSy if 2^n divides x if and only if it divides y.

(i) Is either of these relations antisymmetric?
(ii) Is either an equivalence relation?
(iii) If one is an equivalence relation, describe the equivalence classes in as simple a way as possible.
(iv) If one is antisymmetric, decide if it is a total (i.e. linear) order.

Ex 6. Let R be a relation on X and define $[x]_R := \{y \in X | xRy\}$.

(i) Suppose that R is an equivalence relation. Show that if $[x]_R \cap [y]_R \neq \emptyset$ then $[x]_R = [y]_R$.
(ii) Which properties of an equivalence relation did you use in your proof? Give an example of a relation R that is *not* an equivalence relation (and not the empty relation) but yet satisfies the statement in (i).

Ex 7. Let $f : X \to Y$ be a function, and consider subsets A, B of X and C, D of Y. Are the following statements true or false? Give a proof or a counterexample.

(i) If $A \cup B = X$ then $f(A) \cup f(B) = Y$.

(ii) If $C \cup D = Y$ then $f^{-1}(C) \cup f^{-1}(D) = X$.

(iii) If $A \cap B = \emptyset$ then $f(A) \cap f(B) = \emptyset$.

(iv) If $C \cap D = \emptyset$ then $f^{-1}(C) \cap f^{-1}(D) = \emptyset$.