INTRODUCTION TO HIGHER MATHEMATICS V2000

Practice Midterm, October 2016

For the exam, you will be expected to know all the main definitions as well as some of the most important proofs. No use of written or printed materials, nor of electronic devices, will be permitted during the exam. Problems will be similar to the ones below, and the exam will be of a similar length, but you are not necessarily expected to finish the entire exam in 75 minutes; do as much as you can.

These and other problems will be discussed during the review session on Thursday, October 6.

1. In this question, X is a set.

(i) Define "relation on X."

(ii) Give an example of a set X and a relation R on X that is symmetric and transitive but not reflexive.

(iii) Define a condition – other than reflexivity! – that guarantees that a relation on X that is symmetric and transitive is also necessarily reflexive.

2. The *Law of the Excluded Middle* is the statement $P \vee \neg P$ where P is any statement. The *Law of Non-Contradiction* is the statement $\neg[P \wedge \neg P]$ where P is any statement.

(i) Let P be any statement. Prove both Laws using truth tables. Prove them again by using Boolean calculation.

(ii) Use De Morgan's Laws (and nothing else) to show that the *Law of the Excluded Middle* is equivalent to the *Law of Non-Contradiction*.

(iii) Are these laws intuitively reasonable? Why or why not? (Either answer is acceptable, but the reasoning has to be cogent.)

3. Define a formula

$$P(a, n, b) \ : \ a = nb$$

where n is a variable in $\mathbb{N}$ and a and b are variables in $\mathbb{Z}$. Consider the following statement:

$$(\exists n \in \mathbb{N})(\forall a \in \mathbb{Z})(\exists b \in \mathbb{Z}) \ P(a, n, b).$$

(a) Explain the meaning of the statement in words. Is it true?

(b) Exchange $\exists$ and $\forall$ in the statement:

$$(\forall n \in \mathbb{N})(\exists a \in \mathbb{Z})(\forall b \in \mathbb{Z}) \ P(a, n, b).$$

What does it mean now? Is it true?

(c) Exchange $(\forall a \in \mathbb{Z})$ and $(\exists b \in \mathbb{Z})$. What does it mean? Is it true?

(d) Exchange $(\forall a \in \mathbb{Z})$ and $(\exists n \in \mathbb{N})$. What does it mean now? Is it true?

4. Let X be the set of functions $f : \mathbb{N} \to \mathbb{N}$. Consider the formula in two variables

$$P(f, a) \ : \ f(a) \neq a$$

where the universe of f is X and the universe of a is $\mathbb{N}$. Consider the statement

$$S \ : (\forall a \in \mathbb{N}) P(f, a).$$

(a) Which variable in S is bound? Which is open?

(b) Let $U = X \times \mathbb{N}$ be the universe of P. Find an element of the characteristic set χ_P of P.

(c) Let $Y \subseteq X$ be the subset for which the statement S is true. Find an element of Y.

5. (a) Consider the set $\mathbb{Z}/5\mathbb{Z}$ of congruence classes modulo 5. If $a \in \mathbb{Z}$ denote its congruence class modulo 5 by $[a]$. Let $f : \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ be the function defined by

$$f([a]) = [2a].$$

(i) Show that f is a well-defined function.
(ii) Show that f is injective.
(iii) Show that f is surjective.

(b) Find the last digits of $3^{7^{253}}$ and 7^{253^3}.

6. Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

(a) Show that f is injective.

(b) What is $f^{-1}([-1, 0])$?

(c) Determine the image of f.