MATH V2000: Review for final, May 2016

Here are answers to some of the questions.

Question 1:

(ii) (8 points) *Give a careful proof by induction on n that*

$$\frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \geq 1$.

Base case: $n = 1$ gives $\frac{1}{2} = \frac{1}{2}$, which is clear.

Inductive step: Suppose that $\frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$. We must prove that

$$\frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

But

$$\begin{aligned} \frac{1}{2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &\overset{\text{by Ind Hyp}}{=} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{(n+1)}{(n+2)} \end{aligned}$$

as required.

(iii) (10 points) *Find integers m, n such that $14m + 13n = 7$.*

I seem to specialize in giving you ones of these you can do in your head! If $m = 1$ and $n = -1$ then $14 \cdot 1 + 13 \cdot (-1) = 1$. So take $m = 7, n = -7$. The general method is to use the Euclidean algorithm to find the gcd and then "back solve". There is a better example of this in question 7(ii).

Question 2: *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $L \in \mathbb{R}$.*

(iii) (8 points) *Show that if $\lim\limits_{x\to a} f(x) = L$ then f is bounded near a, i.e. there are constants $C, M > 0$ so that $|f(x)| < M$ for all x such that $0 < |x - a| < C$.*

By definition of limit, if we take $\epsilon = 1$ there is $\delta > 0$ so that $0 < |x - a| < \delta$ implies $|f(x) - L| < 1$. But by the triangle inequality

$$|f(x)| \leq |f(x) - L + L| \leq |f(x) - L| + |L| \leq 1 + |L|.$$

So we may take $C = \delta$ and $M = 1 + |L|$.

Notice that the question was slightly wrong – I wrote the condition on x as $|x - a| < C$ instead of $0 < |x - a| < C$.

Question 3: (iii) (7 points) *If L is Dedekind cut, is the set $\{x^2 : x \in L\}$ a Dedekind cut? What about the set $\{0\} \cup \{\frac{1}{x} : x \in L \setminus 0\}$?*

Neither of these need be a Dedekind cut. For example if $L = (-\infty, -1) \cap \mathbb{Q}$, then the set $\{x^2 : x \in L\}$ consists only of positive numbers and so fails condition (III): for example $4 \in \{x^2 : x \in L\}$ but $-1 < 4$ is not in this set.

Further with this L the set $\{0\} \cup \{\frac{1}{x} : x \in L \setminus 0\}$ is contained in the interval $(-1, 0)$ and does not contain -2.

(In fact there is no L for which these sets are Dedekind cuts.)

Question 4:
(i) (10 points) *Let $m, n \in \mathbb{N}$. Prove by induction on m that if there is a bijection $\lceil m \rceil \to \lceil n \rceil$ then $m = n$.*

This is bookwork – look at the proof on p 152-153 in DM.

(ii)(10 points) *Are there surjections $\mathbb{R} \to \mathbb{N}$ or $\mathbb{N} \to \mathbb{R}$? Explain your answer.*

There is a surjection $\mathbb{R} \to \mathbb{N}$ eg define $f(x) = |\lfloor x \rfloor|$, where $\lfloor x \rfloor|$ is the largest integer that is less than or equal to x.

There is no surjection $\mathbb{N} \to \mathbb{R}$ since $\mathbb{R}$ is uncountable and so has cardinality strictly $>$ that of $|\mathbb{N}|$.

One argument: Identify $\mathbb{R}$ with the set of infinite decimals and suppose that $f : \mathbb{N} \to \mathbb{R}$ is any map. Then define the decimal $x := r_0 \cdot r_1 r_2 r_3 \ldots$ as follows:

$r_0 = 1$ if the number $f(0)$ does not have 1 in the units place, $r_0 = 2$ otherwise,
$r_1 = 1$ if the number $f(1)$ does not have 1 in the first decimal place, $r_1 = 2$ otherwise,
and in general
$r_k = 1$ if the number $f(k)$ does not have 1 in the kth decimal place, $r_k = 2$ otherwise,
Then the real number x is not equal to $f(k)$ for any $k \in \mathbb{N}$. Hence f is NOT surjective.

Question 5: *Let X, Y, Z be any sets and $f : X \to Y$, $g : Y \to Z$ be functions.*
(ii) (5 points) *Show that if $g \circ f$ is injective then so is f.* – bookwork

(iii) (5 points)*If $g \circ f$ is surjective must g be surjective?* YES – you should give a proof.
eg Let $z \in Z$. Since $g \circ f$ is surjective there is $x \in X$ such that $g \circ f(x) = g(f(x)) = z$.
Therefore $g(y) = z$ where $y = f(x)$. Hence g is surjective

(iv) (8 points) *Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x(x-1)(x-2)$; note that $f(3) = 6$.*

- *What is $f^{-1}([0,6])$?* $[0,1] \cup [2,3]$.
- *If $A = [-1,0]$ and $B = [2,3]$ what are $f(A) \cup f(B)$, $f(A) \cap f(B)$?*
  $f(A) \cup f(B) = [-6,6]$ and $f(A) \cap f(B) = \{0\}$.
- *Find two distinct intervals C, D such that $\emptyset \neq f(C) \cap f(D) = f(C \cap D)$.*
  Take $C = [-1,0]$ and $D = [0, \frac{1}{2}]$.

Question 6: (5 points each) *Consider the statement*

$$P : \quad \exists x \in \mathbb{R}, \text{ such that } \forall y \in \mathbb{R}, x^2 > y \Longrightarrow x > y.$$

(i) *Write down its negative in a form that does not involve any negations.*

for all $x \in \mathbb{R}$, there is $y \in \mathbb{R}$ such that $x^2 > y$ and $x \leq y$.

(ii) *Is P true or false ?*
P is true: take $x = 1$. (or anything > 1)

(iv) *Show that $\sqrt{3}$ is an irrational number.*

Argue by contradiction: Assume $\frac{p}{q} = \sqrt{3}$ where $\frac{p}{q}$ is in lowest terms. Thrn $p^2 = 3q^2$. Hence (by Fund Thm of arithmetic) $3|p$, i.e. $p = 3k$. Then $9k^2 = 3q^2$ so $3k^2 = q^2$. So 3 must also divide q, which is a contradiction.

Question 7: (i) (10 points) *Let $a, b \in \mathbb{N}$. Show that* $\gcd(a, b)$ *is the smallest positive element in the set* $\{ma + nb \mid m, n \in \mathbb{Z}\}$*. You may use without proof that if the positive integers a, b are relatively prime, then there are integers m, n such that $ma + nb = 1$, and that if $a > b$ then* $\gcd(a - b, b) = \gcd(a, b)$*. But prove all other results that you use.*

Let $g = \gcd(a, b)$ and $s =$ smallest positive element in the set $\{ma + nb \mid m, n \in \mathbb{Z}\}$. We must show $g \leq s$ and $s \leq g$.

Since $g|a$ and $g|b$, g is a divisor of every element of the form $ma + nb$. Hence $g|s$, Thus $g \leq s$.

Next we show that $\gcd(\frac{a}{g}, \frac{b}{g}) = 1$. But if not, they have a common divisor $d > 1$. But $d|\frac{a}{g}$ implies that $dg|a$. Similarly, $d|\frac{b}{g}$ implies that $dg|b$. Therefore dg divides both a, b. Since g is the largest common divisor and $dg \geq g$ this means that $dg = g$, i.e. $d = 1$. Therefore $\gcd(\frac{a}{g}, \frac{b}{g}) = 1$.

This means that there are m, n so that $m\frac{a}{g} + n\frac{b}{g} = 1$. So, multiplying by g we get $ma + nb = g$. Therefore $g \in \{ma + nb \mid m, n \in \mathbb{Z}\}$. Therefore $s \leq g$ (since s is smallest positive element in this set.)

This completes the proof.

(ii) (10 points) *Find* $c = \gcd(3999, 1419)$ *and find $m, n \in \mathbb{Z}$ so that $3999m + 1419n = c$.*

gcd $= 129$. $m = 5$ and $n = -14$,

Question 8: Are the following true or false? Give reasons for your answers.

(i) (5 points) *Define a relation on the subsets of X by saying $A \; R \; B \iff A \cap B \neq \emptyset$. This is an equivalence relation.*

False This relation is not transitive in general: eg $A \cap C$ could be empty, but both these sets could have nonempty intersection with B. eg for intervals in $\mathbb{R}$:
take $A = [0, 1], B = [1, 2]$ and $C = [2, 3]$.

(ii) (5 points) *Define a relation R on pairs $(a, b) \in \mathbb{N}^+ \times \mathbb{N}^+$ by setting*

$$(a, b)R(c, d) \iff ab \geq cd.$$

This is an order relation. (Here $\mathbb{N}^+ = \{n \in \mathbb{N} : n > 0\}$.)

False This is not antisymmetric. eg $(4, 9)R(18, 18)$ and $(18, 18)R(4.9)$ but $(4, 9) \neq (8, 18)$.

(iii) (5 points) *Given any $m, n \in N$ with* $\gcd(m, n) > 1$*, we can write n uniquely as a product ab where* $\gcd(m, a) = 1$ *and* $\gcd(m, b) > 1$.

False; given m, n there might several decompositions of this kind. eg if $m = 15, n = 66$ we could take $a = 3, b = 22$ or $a = 6, b = 11$.

(iv) (5 points) *If X is uncountable and $f : \mathbb{N} \to X$ is any map, then $|X \setminus f(\mathbb{N})| = |X|$.*

True.

Since there is an obvious injection $X \setminus f(\mathbb{N}) \to X$, by the Sch-Bern theorem we only need show that there is an injection $g : X \to X \setminus f(\mathbb{N})$.

First note that $X \setminus f(\mathbb{N})$ is infinite. (since otherwise X is the union of the two countable sets $f(\mathbb{N})$ and $X \setminus f(\mathbb{N})$.) Therefore there is an injection $h : \mathbb{N} \to X \setminus f(\mathbb{N})$.

Now define $g : X \to X \setminus f(\mathbb{N})$ as follows:

– if $x \in X \setminus (f(\mathbb{N}) \cup h(\mathbb{N}))$, put $g(x) = x$;

– choose an injection ι from the subset $f(\mathbb{N}) \subset X$ onto the odd numbers in $\mathbb{N}$, and if $x \in f(\mathbb{N})$, define $g(x) = h(\iota(x))$;

– if $x \in h(\mathbb{N})$, define $g(x) = h(2h^{-1}(x))$.

Then this is injective since the elements of $f(\mathbb{N})$ map to images of odd numbers under h, while the elements of $h(\mathbb{N})$ map to images of even numbers under h.