INTRODUCTION TO HIGHER MATHEMATICS V2000

Homework, week 11, due December 8

1. Exercises 8.8, 8.9, 8.16, 8.18, 8.20 of Dumas-McCarthy ask for complete proofs of basic properties of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Choose three of them and write up the proofs.

2. We show that Dedekind cuts and decimal expansions give the same theory of the real numbers.

Let $r_1 \leq r_2 \leq r_3 \leq \dots$ be any non decreasing sequence of elements of $\mathbb{Q}$ such that

(a) the sequence is bounded, i.e. $\exists M \in \mathbb{Q}$ such that $r_n < M$ for all $n \in \mathbb{N}$; and

(b) the sequence is not eventually constant, i.e. for all n there is $m > n$ with $r_m > r_n$.

(i) Show that

$$C = \bigcup_{n \geq 1} (-\infty, r_n) \cap \mathbb{Q}$$

is a Dedekind cut.

(ii) Given any infinite decimal that does not eventually end with all 0s, choose r_n as above so that the Dedekind cut corresponds to the decimal.

(iii) Conversely, let C be a Dedekind cut attached to $r_1 \leq r_2 \leq r_3 \leq \dots$ as above. Find a set $s_1 \leq s_2 \leq s_3 \leq \dots$ with properties (a) and (b) as above such that

(a) For all $n > 0$, $s_n = \frac{a_n}{10^n}$ for some integer a_n; in other words, s_n is a truncated decimal; and

(b)

$$C = \bigcup_{n \geq 1} (-\infty, s_n) \cap \mathbb{Q}$$

In other words, given any rational number a that is less than r_n for some n, we can find s_m in the sequence such that $a < s_m$.

3. Prove that any polynomial of the form $x^n + \sum_{i<n} a_i x_i$, with $a_i \in \mathbb{R}$ and n *odd*, has at least one real root.