Problem 13.1.16.

II: expanding spiral, circular in projection to xz-plane

Problem 13.1.21.

II: expanding spiral, circular in projection to xz-plane

Problem 13.1.22.

VI: helix around z-axis with maximum z value 1

Problem 13.1.23.

V: unbounded and parabolic in projection to xz-plane

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I: sinusoidal along $z$, circular in projection to $xy$-plane

Problem 13.1.25.

IV: helix around $z$-axis with unbounded $z$


III: $x + y = 1$ with $x, y \geq 0$

Problem 13.2.6.

We have $\vec{r}(t) = (e^t, 2t)$ and $\vec{r}'(t) = (e^t, 2)$, so $\vec{r}(0) = (1, 0)$ and $\vec{r}'(0) = (1, 2)$. When graphing the function, observe that $x = e^{\frac{z}{2}}$. 

\begin{center}
\includegraphics[width=0.5\textwidth]{graph1.png}
\end{center}

Problem 13.2.7.

We have $\vec{r}(t) = (4 \sin(t), -2 \cos(t))$ and $\vec{r}'(t) = (4 \cos(t), 2 \sin(t))$, so $\vec{r}(\frac{3\pi}{4}) = (2\sqrt{2}, \sqrt{2})$ and $\vec{r}'(\frac{3\pi}{4}) = (-2\sqrt{2}, \sqrt{2})$. When graphing the function, observe that $\frac{x^2}{16} + \frac{y^2}{4} = 1$. 

\begin{center}
\includegraphics[width=0.5\textwidth]{graph2.png}
\end{center}
Problem 13.2.10.

Take the derivative component-wise, so \( \vec{r}'(t) = (-e^{-t}, t - 3t^2, \frac{1}{t}) \).

Problem 13.2.11.

Take the derivative component-wise, so \( \vec{r}''(t) = (2t, -2t \sin(t^2), \frac{1}{t}) \).

Problem 13.2.12.

Take the derivative component-wise, so \( \vec{r}''(t) = \left( -\frac{1}{(1+t)^3}, \frac{1}{(1+t)^2}, \frac{t^3 + 2t}{(1+t)^3} \right) \). Be sure to simplify!

Problem 13.2.16.

We have \( \vec{r}'(t) = \frac{d}{dt}(t \vec{a}) \times (\vec{b} + t \vec{c}) + t \vec{a} \times \frac{d}{dt}(\vec{b} + t \vec{c}) = \vec{a} \times (\vec{b} + t \vec{c}) + t \vec{a} \times \vec{c} = \vec{a} \times (\vec{b} + 2t \vec{c}) \).

Problem 13.2.21.

Since \( \vec{r}(t) = (t, t^2, t^3) \), we have \( \vec{r}'(t) = (1, 2t, 3t^2) \) and \( \vec{r}''(t) = (1, 2, 6t) \). Then \( \vec{r}'(1) = (1, 2, 3), \vec{r}(1) = \frac{\vec{r}'(1)}{||\vec{r}'(1)||} = \left( \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) \), and \( \vec{r}'(t) \times \vec{r}''(t) = (6t^2, -6t, 2) \).


We have \( \vec{r}'(t) = \left( \frac{1}{\sqrt{t^2+1}}, \frac{2t}{t^2+1}, 1 \right) \) so \( \vec{r}'(1) = \left( \frac{1}{2}, \frac{1}{2}, 1 \right) \). Furthermore, \( \vec{r}(1) = (2, \ln(4), 1) \) so the tangent line at \( (2, \ln(4), 1) \), i.e. \( t = 1 \), is \( \ell : \left( \frac{1}{2} + 2, \frac{1}{2} + \ln(4), t + 1 \right) \).

Problem 13.2.27.

If we let \( x = 5 \cos(t), y = 5 \sin(t) \), then we have \( 25 \sin^2(t) + z^2 = 20 \), so \( z = \pm \sqrt{20 - 25 \sin^2(t)} \). Then writing \( \vec{r}(t) = (5 \cos(t), 5 \sin(t), \sqrt{20 - 25 \sin^2(t)}) \), we have \( \vec{r}'(t) = (-5 \sin(t), 5 \cos(t), -\frac{50 \sin(t) \cos(t)}{\sqrt{20 - 25 \sin^2(t)}}) \). At point \( \vec{r}(t_0) = (3, 4, 2) \), we have that \( 3 = 5 \cos(t_0) \) and \( 4 = 5 \sin(t_0) \), so \( \cos(t_0) = \frac{3}{5} \) and \( \sin(t_0) = \frac{4}{5} \). Then \( \vec{r}'(t_0) = (-4, 3, -6) \). Then \( \ell \) is given by \( (3 - 4t, 4 + 3t, 2 - 6t) \).

Note that we can use any multiple of \( (-4, 3, -6) \) for the coefficient of \( t \). For instance, if we set \( t := x \), then we would arrive at \( \ell : (3 + t, 4 - \frac{3}{5}t, 2 + \frac{3}{5}t) \). If we set \( t := y \), then we would have \( \ell : (3 - \frac{4}{5}t, 4 + t, 2 - 2t) \).

Problem 13.2.34.
Calculating the intersection of $\vec{r}_1(t)$ and $\vec{r}_2(s)$, we see that $t = 3 - s$ and $3 + t^2 = s^2$, so $s^2 = 3 + (3 - s)^2 = 12 - 6s + s^2$ from which we can deduce $s = 2$ and $t = 1$. Then $\vec{r}_1(t)$ and $\vec{r}_2(s)$ intersect at $\vec{r}_1(1) = \vec{r}_2(2) = (1,0,4)$.

We can calculate $\vec{r}_1'(t) = (1,-1,2t)$ so $\vec{r}_1'(1) = (1,-1,2)$. Similarly, $\vec{r}_2'(s) = (-1,1,2s)$ so $\vec{r}_2'(2) = (-1,1,4)$. Then

$$\theta = \cos^{-1}\left(\frac{\vec{r}_1'(1) \cdot \vec{r}_2'(2)}{||\vec{r}_1'(1)|| ||\vec{r}_2'(2)||}\right) = \cos^{-1}\left(\frac{6}{\sqrt{6\sqrt{18}}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

This is approximately $54.7^\circ$, but the expression $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is the “real” (exact) answer!

**Problem 13.2.37.**

We just evaluate the integral component-wise, so we have

$$\left(\int_{0}^{1} \frac{1}{t+1} \, dt, \int_{0}^{1} \frac{1}{t^2+1} \, dt, \int_{0}^{1} \frac{t}{t^2+1} \, dt\right) = \left(\ln|t+1|\bigg|_{0}^{1}, \tan^{-1}(t)\bigg|_{0}^{1}, \frac{1}{2} \ln|t^2+1|\bigg|_{0}^{1}\right) = \left(\ln(2), \frac{\pi}{4}, \frac{1}{2} \ln(2)\right).$$

**Problem 13.2.42.**

Integrating component-wise, we have $\vec{r}(t) = (\frac{t^2}{2} + A, e^t + C, e^t(t - 1) + C)$ and $\vec{r}(0) = (1,1,1)$. Then we must have $A = 1, B = 0, C = 2$, i.e. $\vec{r}(t) = (\frac{t^2}{2} + 1, e^t, e^t(t - 1) + 2)$.

**Problem 13.3.1.**

First, compute $\vec{r}'(t) = (1,-3 \sin(t), 3 \cos(t))$. Then the length of the curve from $t = -5$ to $t = 5$ is given by

$$L = \int_{-5}^{5} ||\vec{r}'(t)|| \, dt = \int_{-5}^{5} \sqrt{1^2 + 9 \sin^2(t) + 9 \cos^2(t)} \, dt = \int_{-5}^{5} \sqrt{1 + 9(\sin^2(t) + \cos^2(t))} \, dt = \int_{-5}^{5} \sqrt{10} \, dt = \sqrt{10}t\bigg|_{-5}^{5} = 10\sqrt{10}.$$
First, compute \( \vec{r}'(t) = (-\sin(t), \cos(t), -\tan(t)) \). Then the length of the curve from \( t = 0 \) to \( t = \frac{\pi}{4} \) is given by

\[
L = \int_0^{\frac{\pi}{4}} |\vec{r}'(t)|\,dt = \int_0^{\frac{\pi}{4}} \sqrt{\sin^2 + \cos^2(t) + \tan^2(t)}\,dt
\]

\[
= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(t)}\,dt = \int_0^{\frac{\pi}{4}} |\sec(t)|\,dt = \ln|\sec(t) + \tan(t)|_{0}^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1).
\]

**Problem 13.3.11.**

First, to parametrize the curve let us set \( x = t \). Then \( y = \frac{x^2}{2} = \frac{t^2}{2} \) and \( z = \frac{xy}{3} = \frac{t^3}{6} \).

Then \( \vec{r}'(t) = (1, t, \frac{t^2}{2}) \), so the length of the curve from \( t = 0 \) to \( t = 6 \) (when \( \vec{r}(t) = (6, 18, 36) \)) is

\[
L = \int_0^{6} |\vec{r}'(t)|\,dt = \int_0^{6} \sqrt{1 + t^2 + \frac{t^4}{4}}\,dt
\]

\[
= \int_0^{6} \sqrt{1 + \frac{t^2}{2}}^2\,dt = \int_0^{6} \left(1 + \frac{t^2}{2}\right)\,dt = \left(t + \frac{t^3}{6}\right)_{0}^{6} = 42.
\]

**Problem 13.3.13.**

Calculate \( \vec{r}'(t) = (-1, 4, 3) \). The point \( P \) has \( 4 = x = 5 - t \), so \( t = 1 \). Then

\[
s(t) = \int_1^{t} ||\vec{r}'(\tau)||\,d\tau = \int_1^{t} \sqrt{26}d\tau = \sqrt{26}t \mid_{1}^{t} = \sqrt{26}(t - 1).
\]

Solving \( s(t) = \sqrt{26}(t - 1) \) for \( t \), we have \( t = \frac{s + \sqrt{26}}{\sqrt{26}} + 1 \). Then \( \vec{r}(s) = (5 - \frac{s}{\sqrt{26}} - 1, 4 \frac{s}{\sqrt{26}} + 4 - 3, 3 \frac{s}{\sqrt{26}} + 3) = (4 - \frac{s}{\sqrt{26}}, 4 \frac{s}{\sqrt{26}} + 1, 3 \frac{s}{\sqrt{26}} + 3) \).

Then at \( s = 4 \), \( \vec{r}(4) = (4 - \frac{4}{\sqrt{26}}, \frac{16}{\sqrt{26}} + 1, \frac{12}{\sqrt{26}} + 3) \).