24. The vector $-5 \mathbf{i} + 3 \mathbf{j} - \mathbf{k}$ has length 
$$| -5 \mathbf{i} + 3 \mathbf{j} - \mathbf{k} | = \sqrt{(-5)^2 + 3^2 + (-1)^2} = \sqrt{35},$$
so by Equation 4 the unit vector with the same direction is 
$$\frac{1}{\sqrt{35}} (-5 \mathbf{i} + 3 \mathbf{j} - \mathbf{k}) = -\frac{5}{\sqrt{35}} \mathbf{i} + \frac{3}{\sqrt{35}} \mathbf{j} - \frac{1}{\sqrt{35}} \mathbf{k}.$$
From the figure we see that \( \tan \theta = \frac{6}{8} = \frac{3}{4} \), so \( \theta = \tan^{-1} \left( \frac{3}{4} \right) \approx 36.9^\circ \).
\[ \overrightarrow{AC} = \frac{1}{3} \overrightarrow{AB} \text{ and } \overrightarrow{BC} = \frac{2}{3} \overrightarrow{BA}. \quad \mathbf{c} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{1}{3} \overrightarrow{AB} \quad \Rightarrow \quad \overrightarrow{AB} = 3 \mathbf{c} - 3 \mathbf{a}. \quad \mathbf{c} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OA} + \frac{2}{3} \overrightarrow{BA} \quad \Rightarrow \quad \overrightarrow{BA} = \frac{3}{2} \mathbf{c} - \frac{3}{2} \mathbf{b}. \quad \overrightarrow{BA} = -\overrightarrow{AB}, \text{ so } \frac{3}{2} \mathbf{c} - \frac{3}{2} \mathbf{b} = 3 \mathbf{a} - 3 \mathbf{c} \iff \mathbf{c} + 2 \mathbf{c} = 2 \mathbf{a} + \mathbf{b} \iff \mathbf{c} = \frac{2}{3} \mathbf{a} + \frac{1}{3} \mathbf{b}. \]
46. Draw \(a\), \(b\), and \(c\) emanating from the origin. Extend \(a\) and \(b\) to form lines \(A\) and \(B\), and draw lines \(A'\) and \(B'\) parallel to these two lines through the terminal point of \(c\). Since \(a\) and \(b\) are not parallel, \(A\) and \(B'\) must meet (at \(P\)), and \(A'\) and \(B\) must also meet (at \(Q\)). Now we see that \(\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{c}\), so if

\[
s = \frac{\overrightarrow{OP}}{|a|} \quad \text{(or its negative, if \(a\) points in the direction opposite \(\overrightarrow{OP}\))}
\]

and

\[
t = \frac{\overrightarrow{OQ}}{|b|} \quad \text{(or its negative, as in the diagram)},
\]

then \(c = sa + tb\), as required.

Argument using components: Since \(a\), \(b\), and \(c\) all lie in the same plane, we can consider them to be vectors in two dimensions. Let \(a = \langle a_1, a_2\rangle\), \(b = \langle b_1, b_2\rangle\), and \(c = \langle c_1, c_2\rangle\). We need \(sa_1 + tb_1 = c_1\) and \(sa_2 + tb_2 = c_2\). Multiplying the first equation by \(a_2\) and the second by \(a_1\) and subtracting, we get

\[
t = \frac{c_2a_1 - c_1a_2}{b_2a_1 - b_1a_2}.
\]

Similarly \(s = \frac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}\).

Since \(a \neq 0\) and \(b \neq 0\) and \(a\) is not a scalar multiple of \(b\), the denominator is not zero.
1. (a) $a \cdot b$ is a scalar, and the dot product is defined only for vectors, so $(a \cdot b) \cdot c$ has no meaning.

(b) $(a \cdot b) c$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|a|$ and $b \cdot c$ are scalars, so $|a| (b \cdot c)$ is an ordinary product of real numbers, and has meaning.

(d) Both $a$ and $b + c$ are vectors, so the dot product $a \cdot (b + c)$ has meaning.

(e) $a \cdot b$ is a scalar, but $c$ is a vector, and so the two quantities cannot be added and $a \cdot b + c$ has no meaning.

(f) $|a|$ is a scalar, and the dot product is defined only for vectors, so $|a| \cdot (b + c)$ has no meaning.
\[ 2 \mathbf{a} \cdot \mathbf{b} = (5, -2) \cdot (3, 4) = (5)(3) + (-2)(4) = 15 - 8 = 7 \]
5. $a \cdot b = \langle 4, 1, \frac{1}{4} \rangle \cdot \langle 6, -3, -8 \rangle = (4)(6) + (1)(-3) + \left(\frac{1}{4}\right)(-8) = 19$
(3i + 2j + k) \cdot (4i + 5k) = (3)(4) + (2)(0) + (1)(5) = 7

9. By Theorem 3, 
\[ |\mathbf{a} \cdot \mathbf{b}| \leq ||\mathbf{a}|| ||\mathbf{b}|| \]
\[ 7 \leq \sqrt{10} \cdot \sqrt{53} \]
\[ 7 \leq 2 \cdot 7.28 \]
\[ 7 \leq 14.56 \]

Thus, the condition of Theorem 3 holds.
17. \(|a| = \sqrt{12 + (-4)^2 + 1^2} = \sqrt{18} = 3\sqrt{2}, \quad |b| = \sqrt{0^2 + 2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}, \) and

\[ a \cdot b = (1)(0) + (-4)(2) + (1)(-2) = -10. \]

From Corollary 6, we have \( \cos \theta = \frac{a \cdot b}{|a| \cdot |b|} = \frac{-10}{3\sqrt{2} \cdot 2\sqrt{2}} = -\frac{10}{12} = -\frac{5}{6}. \)

the angle between \(a\) and \(b\) is \( \theta = \cos^{-1} \left( -\frac{5}{6} \right) \approx 146^\circ. \)
23. (a) \( \mathbf{a} \cdot \mathbf{b} = (9)(-2) + (3)(6) = 0 \), so \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal (and not parallel).

(b) \( \mathbf{a} \cdot \mathbf{b} = (4)(3) + (5)(-1) + (-2)(5) = -3 \neq 0 \), so \( \mathbf{a} \) and \( \mathbf{b} \) are not orthogonal. Also, since \( \mathbf{a} \) is not a scalar multiple of \( \mathbf{b} \), \( \mathbf{a} \) and \( \mathbf{b} \) are not parallel.

(c) \( \mathbf{a} \cdot \mathbf{b} = (-8)(6) + (12)(-9) + (4)(-3) = -168 \neq 0 \), so \( \mathbf{a} \) and \( \mathbf{b} \) are not orthogonal. Because \( \mathbf{a} = -\frac{4}{3} \mathbf{b} \), \( \mathbf{a} \) and \( \mathbf{b} \) are parallel.

(d) \( \mathbf{a} \cdot \mathbf{b} = (3)(5) + (-1)(9) + (3)(-2) = 0 \), so \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal (and not parallel).
31. The curves \( y = x^2 \) and \( y = x^3 \) meet when \( x^2 = x^3 \Leftrightarrow x^3 - x^2 = 0 \Leftrightarrow x^2(x - 1) = 0 \Leftrightarrow x = 0, x = 1 \). We have
\[
\frac{d}{dx} x^2 = 2x \quad \text{and} \quad \frac{d}{dx} x^3 = 3x^2,
\]
so the tangent lines of both curves have slope 0 at \( x = 0 \). Thus the angle between the curves is \( 0^\circ \) at the point \((0, 0)\).

For \( x = 1 \),
\[
\frac{d}{dx} x^2 \bigg|_{x=1} = 2 \quad \text{and} \quad \frac{d}{dx} x^3 \bigg|_{x=1} = 3
\]
so the tangent lines at the point \((1, 1)\) have slopes 2 and 3.

3. Vectors parallel to the tangent lines are \( \langle 1, 2 \rangle \) and \( \langle 1, 3 \rangle \), and the angle \( \theta \) between them is given by
\[
\cos \theta = \frac{\langle 1, 2 \rangle \cdot \langle 1, 3 \rangle}{||\langle 1, 2 \rangle|| \cdot ||\langle 1, 3 \rangle||} = \frac{1 + 6}{\sqrt{5} \sqrt{10}} = \frac{7}{5\sqrt{2}}
\]
Thus \( \theta = \cos^{-1} \left( \frac{7}{5\sqrt{2}} \right) \approx 8.1^\circ \).
43. \(|a| = \sqrt{9 + 9 + 1} = \sqrt{19}\) so the scalar projection of \(b\) onto \(a\) is 
\[\text{comp}_a b = \frac{a \cdot b}{|a|} = \frac{6 - 12 - 1}{\sqrt{19}} = \frac{-7}{\sqrt{19}}.\]

While the vector projection of \(b\) onto \(a\) is 
\[\text{proj}_a b = -\frac{7}{\sqrt{19}} \frac{a}{|a|} = -\frac{7}{\sqrt{19}} \frac{1}{\sqrt{19}} (3i - 3j + k) = -\frac{7}{19} (3i - 3j + k) = -\frac{21}{19} i + \frac{21}{19} j - \frac{7}{19} k.\]
54. \((r - a) \cdot (r - b) = 0\) implies that the vectors \(r - a\) and \(r - b\) are orthogonal.

From the diagram (in which \(A\), \(B\) and \(R\) are the terminal points of the vectors), we see that this implies that \(R\) lies on a sphere whose diameter is the line from \(A\) to \(B\). The center of this circle is the midpoint of \(AB\), that is, 
\[
\frac{1}{2}(a + b) = \left(\frac{1}{2} (a_1 + b_1), \frac{1}{2} (a_2 + b_2), \frac{1}{2} (a_3 + b_3)\right),
\]
and its radius is 
\[
\frac{1}{2} |a - b| = \frac{1}{2} \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.
\]

Or: Expand the given equation, substitute \(r \cdot r = x^2 + y^2 + z^2\) and complete the squares.
64. If the vectors $u + v$ and $u - v$ are orthogonal then $(u + v) \cdot (u - v) = 0$. But

\[
(u + v) \cdot (u - v) = (u + v) \cdot u - (u + v) \cdot v
\]

\[
= u \cdot u + v \cdot u - u \cdot v - v \cdot v
\]

\[
= |u|^2 + u \cdot v - u \cdot v - |v|^2
\]

\[
= |u|^2 - |v|^2
\]

by Property 3 of the dot product

by Property 3

by Properties 1 and 2

Thus $|u|^2 - |v|^2 = 0 \Rightarrow |u|^2 = |v|^2 \Rightarrow |u| = |v|$ [since $|u|, |v| \geq 0$].
\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
3 & 3 & -3 \\
3 & -3 & 3 \\
\end{vmatrix} = \begin{vmatrix}
3 & -3 & \mathbf{i} - \begin{vmatrix}
3 & -3 & \mathbf{j} \\
3 & 3 & \mathbf{k} \\
\end{vmatrix}
\end{vmatrix}
\]

\[
= (9 - 9) \mathbf{i} - [9 - (-9)] \mathbf{j} + (-9 - 9) \mathbf{k} = -18 \mathbf{j} - 18 \mathbf{k}
\]

Since \((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (-18 \mathbf{j} - 18 \mathbf{k}) \cdot (3 \mathbf{i} + 3 \mathbf{j} - 3 \mathbf{k}) = 0 - 54 + 54 = 0\), \(\mathbf{a} \times \mathbf{b}\) is orthogonal to \(\mathbf{a}\).

Since \((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (-18 \mathbf{j} - 18 \mathbf{k}) \cdot (3 \mathbf{i} - 3 \mathbf{j} + 3 \mathbf{k}) = 0 + 54 - 54 = 0\), \(\mathbf{a} \times \mathbf{b}\) is orthogonal to \(\mathbf{b}\).
6. \( \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} i - \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} j + \begin{vmatrix} t & \cos t \\ 1 & -\sin t \end{vmatrix} k \\
= [\cos^2 t - (-\sin^2 t)] i - (t \cos t - \sin t) j + (-t \sin t - \cos t) k = \mathbf{i} + (\sin t - t \cos t) \mathbf{j} + (-t \sin t - \cos t) \mathbf{k} \)

Since

\[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = [\mathbf{i} + (\sin t - t \cos t) \mathbf{j} + (-t \sin t - \cos t) \mathbf{k}] \cdot (t \mathbf{i} + \cos t \mathbf{j} + \sin t \mathbf{k}) \]
\[= t + \sin t \cos t - t \cos^2 t - t \sin^2 t - \sin t \cos t \]
\[= t - t (\cos^2 t + \sin^2 t) = 0 \]

\( \mathbf{a} \times \mathbf{b} \) is orthogonal to \( \mathbf{a} \).

Since

\[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = [\mathbf{i} + (\sin t - t \cos t) \mathbf{j} + (-t \sin t - \cos t) \mathbf{k}] \cdot (\mathbf{i} - \sin t \mathbf{j} + \cos t \mathbf{k}) \]
\[= 1 - \sin^2 t + t \sin t \cos t - t \sin t \cos t - \cos^2 t \]
\[= 1 - (\sin^2 t + \cos^2 t) = 0 \]

\( \mathbf{a} \times \mathbf{b} \) is orthogonal to \( \mathbf{b} \).
13. (a) Since $\mathbf{b} \times \mathbf{c}$ is a vector, the dot product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b} \cdot \mathbf{c}$ is a scalar, so $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ is meaningless, as the cross product is defined only for two vectors.

(c) Since $\mathbf{b} \times \mathbf{c}$ is a vector, the cross product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is meaningful and results in another vector.

(d) $\mathbf{b} \cdot \mathbf{c}$ is a scalar, so the dot product $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ is meaningless, as the dot product is defined only for two vectors.

(e) Since $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{c} \cdot \mathbf{d})$ are both scalars, the cross product $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ is meaningless.

(f) $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are both vectors, so the dot product $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ is meaningful and is a scalar.
14. Using Theorem 9, we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta = (4)(5) \sin 45^\circ = 20 \cdot \frac{\sqrt{2}}{2} = 10 \sqrt{2}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.
25. \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times (b_1 + c_1, b_2 + c_2, b_3 + c_3) \)
\[ = \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle \]
\[ = \langle a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1 \rangle \]
\[ = \langle (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2), (a_3b_1 - a_1b_3) + (a_3c_1 - a_1c_3), (a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1) \rangle \]
\[ = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1 \rangle \]
\[ = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) \]
The distance between a point and a line is the length of the perpendicular from the point to the line, here \( |\vec{PS}| = d \). But referring to triangle \( PQS \),

\[
d = |\vec{PS}| = |\vec{QP}| \sin \theta = |\mathbf{b}| \sin \theta.
\]

But \( \theta \) is the angle between \( \vec{QP} = \mathbf{b} \) and \( \vec{QR} = \mathbf{a} \). Thus by Theorem 9, \( \sin \theta = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} \)

and so

\[
d = |\mathbf{b}| \sin \theta = \frac{|\mathbf{b}| |\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}.
\]

(b) \( \mathbf{a} = \vec{QR} = (-1, -2, -1) \) and \( \mathbf{b} = \vec{QP} = (1, -5, -7) \). Then

\[
\mathbf{a} \times \mathbf{b} = \langle (-2)(-7) - (-1)(-5), (-1)(1) - (-1)(-7), (-1)(-5) - (-2)(1) \rangle = \langle 9, -8, 7 \rangle.
\]

Thus the distance is

\[
d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{1}{\sqrt{6}} \sqrt{81 + 64 + 49} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}.
\]
53. (a) No. If \( \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \), then \( \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0 \), so \( \mathbf{a} \) is perpendicular to \( \mathbf{b} - \mathbf{c} \), which can happen if \( \mathbf{b} \neq \mathbf{c} \). For example, let \( \mathbf{a} = \langle 1, 1, 1 \rangle \), \( \mathbf{b} = \langle 1, 0, 0 \rangle \) and \( \mathbf{c} = \langle 0, 1, 0 \rangle \).

(b) No. If \( \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \) then \( \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0 \), which implies that \( \mathbf{a} \) is parallel to \( \mathbf{b} - \mathbf{c} \), which of course can happen if \( \mathbf{b} \neq \mathbf{c} \).

(c) Yes. Since \( \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \), \( \mathbf{a} \) is perpendicular to \( \mathbf{b} - \mathbf{c} \), by part (a). From part (b), \( \mathbf{a} \) is also parallel to \( \mathbf{b} - \mathbf{c} \). Thus since \( \mathbf{a} \neq \mathbf{0} \) but is both parallel and perpendicular to \( \mathbf{b} - \mathbf{c} \), we have \( \mathbf{b} - \mathbf{c} = \mathbf{0} \), so \( \mathbf{b} = \mathbf{c} \).
2. (a) 

\[ x = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2} = 2 \cdot 1 \cdot 0 = 0, \]
\[ y = 2 \sin \frac{\pi}{2} \sin \frac{\pi}{2} = 2 \cdot 1 \cdot 1 = 2, \]
\[ z = 2 \cos \frac{\pi}{2} = 2 \cdot 0 = 0 \]

so the point is \((0, 2, 0)\) in rectangular coordinates.

(b) 

\[ x = 4 \sin \frac{\pi}{3} \cos \left( -\frac{\pi}{4} \right) = 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \sqrt{6}, \]
\[ y = 4 \sin \frac{\pi}{3} \sin \left( -\frac{\pi}{4} \right) = 4 \left( \frac{\sqrt{3}}{2} \right) \left( -\frac{\sqrt{2}}{2} \right) = -\sqrt{6}, \]
\[ z = 4 \cos \frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2 \]

so the point is \((\sqrt{6}, -\sqrt{6}, 2)\) in rectangular coordinates.
4. (a) \( \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 0 + 3} = 2, \cos \phi = \frac{z}{\rho} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}, \) and \( \cos \theta = \frac{x}{\rho \sin \phi} = \frac{1}{2 \sin(\pi/6)} = 1 \Rightarrow \theta = 0. \) Thus spherical coordinates are \((2, 0, \frac{\pi}{6})\).

(b) \( \rho = \sqrt{3^2 + 1^2 + 2^2} = 4, \cos \phi = \frac{z}{\rho} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}, \) and \( \cos \theta = \frac{x}{\rho \sin \phi} = \frac{\sqrt{3}}{4 \sin(\pi/6)} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{11\pi}{6} \) [since \( y < 0 \)]. Thus spherical coordinates are \((4, \frac{11\pi}{6}, \frac{\pi}{6})\).
7. From Equations 1 we have $z = \rho \cos \phi$, so $\rho \cos \phi = 1 \Leftrightarrow z = 1$, and the surface is the horizontal plane $z = 1$.
\[ \rho = \cos \phi \implies \rho^2 = \rho \cos \phi \iff x^2 + y^2 + z^2 = z \iff x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4} \iff x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}. \]

Therefore, the surface is a sphere of radius \( \frac{1}{2} \) centered at \((0, 0, \frac{1}{2})\).