Homework, WEEK 2

1.25 $X = \{x_1, x_2, \dots, x_n\}$

1) $n > 0$ # of permutations of $X = n! = n(n-1)(n-2)\cdots 1$

2) $n = 0$ # of permutations $= 1$

1.29

ii) $f_2(X) = X^3 - X + 2$ Let the interval X be $(1, 2)$

Then, $f[X] = (2, 8)$ and $f: X \to f[X]$ is a bijection.

Formula for the inverse function (Implicit formula is given)

$$f^{-1}(y) = X \text{ such that } X^3 - X + 2 = y \text{ and } y \in (2, 8)$$

or $f^{-1}(y) = \{X \mid X^3 - X + 2 = y, \text{ and } y \in (2, 8)\}$

iii) $f_3(X) = \dfrac{X^2+1}{X^2-1}$ Let X be $(2, 3)$, then $f[X] = \left(\frac{5}{4}, \frac{5}{3}\right)$

$f_3(X) = \dfrac{X^2 - 1 + 2}{X^2 - 1}$ To Find the Formula for the inverse function:

$$= 1 + \frac{2}{(X+1)(X-1)}$$

$$= 1 + \left[\frac{1}{X+1} - \frac{1}{X-1}\right]\cdot(-1)$$

$$= 1 + \frac{1}{X-1} - \frac{1}{X+1}$$

$$y = \frac{X^2+1}{X^2-1} = 1 + \frac{2}{X^2-1}$$

$$\frac{Y-1}{2} = \frac{1}{X^2-1}$$

$$\frac{2}{Y-1} = X^2 - 1 \text{ so } X^2 = \frac{2+Y-1}{Y-1} = \frac{Y+1}{Y-1}$$

Since $Y \in \left(\frac{5}{4}, \frac{5}{3}\right)$, $\dfrac{Y+1}{Y-1} > 0$

Also, since $f^{-1}(y) \in (2, 3)$, $f^{-1}(y) > 0$. Thus $X = \sqrt{\dfrac{Y+1}{Y-1}}$

and $f^{-1}(y) = \sqrt{\dfrac{Y+1}{Y-1}}$ for $y \in \left(\frac{5}{4}, \frac{5}{3}\right)$ is the inverse function.

1.31 For $b \in \mathbb{R}$, either $f^{-1}(b)$ is empty or it's not empty
If $f^{-1}(b)$ is not empty, then, Let $A = \{a \mid f(a) = b\}$ is not empty.
For any $a_1, a_2 \in A$, we have $f(a_1) = b$, $f(a_2) = b$. Therefore, $f(a_1) = f(a_2) = b$. $(*)$
Since f is strictly increasing, if $a_1 \neq a_2$, then ~~either~~ $f(a_1) - f(a_2) \neq 0$.
In $(*)$, we see that $f(a_1) = f(a_2)$. Therefore, $a_1 = a_2$. Thus, $f^{-1}(b) = A$ has a single element if it's not empty, and thus f is an injection.
If f is a bijection, the inverse of f is also strictly increasing.
Let $b_1, b_2 \in \mathbb{R}$ and $b_1 < b_2$. Let $a_1 = f^{-1}(b_1)$ and $a_2 = f^{-1}(b_2)$. $(**)$
Since f is strictly increasing, $f(x) < f(y)$ for all $x, y \in \mathbb{R}$ and $x < y$.
If $a_1 \geq a_2$, then $f(a_1) = b_1 \geq b_2 = f(a_2)$, Contradicts $(**)$
Therefore, we conclude that $a_1 < a_2$
Since for any $b_1, b_2 \in \mathbb{R}$ $b_1 < b_2 \Rightarrow f^{-1}(b_1) < f^{-1}(b_2)$.
f^{-1} is strictly increasing.

1.33

(iii) $\bigcap_{k=1}^{5} [\bigcup_{n=1}^{k} X_n]$

$X_1 = \{2\}$, $X_2 = \{3,4\}$, $X_3 = \{4,5,6\}$, $X_4 = \{5,6,7,8\}$ $X_5 = \{6,7,8,9,10\}$

so $\bigcap_{k=1}^{5} [\bigcup_{n=1}^{k} X_n] = X_1 \cap \{\bigcup_{n=1}^{2} X_n\} \cap \{\bigcup_{n=1}^{3} X_n\} \cap \dots \cap \{\bigcup_{n=1}^{5} X_n\}$

$= \{2\} \cap \{2,3,4\} \cap \{2,3,4,5,6\} \cap \dots \cap \{2,3,4,\dots,10\}$

$= \{2\}$

(iv) $\bigcap_{k=5}^{\infty} [\bigcup_{n=3}^{k} X_n] = \{4,5,6,7,8,9,10\} \cap \{4,5,6,7,8,9,10,11,12\} \cap \dots$

$= \{4,5,6,7,8,9,10\}$

1.35 (i) $f: X \to Y$

Let $y \in Y$. 1) If $y \in f(\bigcup_{\alpha \in A} U_\alpha)$, then, $\exists x_0 \in \bigcup_{\alpha \in A} U_\alpha$ s.t. $f(x_0) = y$. Since $x_0 \in \bigcup_{\alpha \in A} U_\alpha$, $x_0 \in U_{\alpha_0}$ for some $\alpha_0 \in A$. Therefore, $y = f(x_0) \in f(U_{\alpha_0})$ and thus, $y \in \bigcup_{\alpha \in A} f(U_\alpha)$

As a result, $f(\bigcup_{\alpha \in A} U_\alpha) \subset \bigcup_{\alpha \in A} f(U_\alpha)$.

2) If $y \in \bigcup_{\alpha \in A} f(U_\alpha)$, then $y \in f(U_{\alpha_0})$ for some $\alpha_0 \in A$.

So, $\exists x_0 \in U_{\alpha_0}$ s.t. $f(x_0) = y$. Since $x_0 \in U_{\alpha_0}$, we have the result that $x_0 \in (\bigcup_{\alpha \in A} U_\alpha)$, and thus $y = f(x_0) \in f(\bigcup_{\alpha \in A} U_\alpha)$.

and $\bigcup_{\alpha \in A} f(U_\alpha) \subset f(\bigcup_{\alpha \in A} U_\alpha)$.

To conclude, $f(\bigcup_{\alpha \in A} U_\alpha) = \bigcup_{\alpha \in A} f(U_\alpha)$

2.9 (i) It has the properties of reflexivity, symmetry

(ii) It has the properties of symmetry,

(iii) It has the properties of reflexivity, antisymmetry and transitivity

(iv) It has the properties of reflexivity, symmetry and transitivity

2.11 $X = \{a, b\}$ Relations on X:

$R_1 = \{(a,b), (b,a), (a,a), (b,b)\}$ $R_2 = \{(a,a), (b,b)\}$, $R_3 = \{(a,b), (b,a)\}$

$R_4 = \{(a,a), (b,a)\}$ $R_5 = \{(a,a), (a,b)\}$ $R_6 = \{(b,a), (b,b)\}$ $R_7 = \{(a,b), (b,b)\}$

$R_8 = \{(a,b), (b,a), (a,a)\}$ $R_9 = \{(a,b), (b,a), (b,b)\}$ $R_{10} = \{(a,b), (a,a), (b,b)\}$

$R_{11} = \{(b,a), (a,a), (b,b)\}$ $R_{12} = \{(a,b)\}$ $R_{13} = \{(b,a)\}$ $R_{14} = \{(a,a)\}$ $R_{15} = \{(b,b)\}$

$R_{16} = \emptyset$

Symmetric: $R_1, R_2, R_3, R_8, R_9, R_{14}, R_{15}, R_{16}$

Reflexive: R_1, R_2, R_{10}, R_{11}

Anti-symmetric: $R_2, R_4, R_5, R_6, R_7, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}$

Transitive: $R_1, R_2, R_4, R_5, R_6, R_7, R_8, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}$

3. Let R_1 be the relation "is contained in"
Let R_2 be the relation " contains"
Let R_3 be the relation is the capital of

$X = \{$ cities, states, lakes $\}$

R_1 has reflexivity, antisymmetry, transitivity

R_2 has reflexivity, antisymmetry, transitivity

R_3 has antisymmetry, transitivity