Solution to Homework 11

8.16

Since all elements of X are less than the elements of Y, we know that X is bounded above and Y is bounded below.

Also, X and Y are non empty sets of real numbers.

(**) According to the least upper bound property, any non-empty set of real numbers that has an upper bound must have a least upper bound in real numbers. (There is a corresponding version for greatest lower bound).

(Think about it. Why is the above version of the least upper bound property the same as the one with Dedekind cuts given in the textbook? Is a non empty set of real numbers a Dedekind cut?)

Therefore, X has a least upper bound and Y has a greatest lower bound. We can denote them by c and d.

If $c = d$, we can let $a = c = d = (c+d)/2$ and thus $x<=a<=y$.

if $c < d$, we can let $a = (c+d)/2$. Therefore, a is greater than c, which is greater than or equal to any element in X.
a is less than d, which is less than or equal to any element in Y.

The case that $c > d$ is impossible because there will be a contradiction(Think about it).

8.18

Check the proof for the existence of the "least upper bound" for a set in **R** that is bounded above in the book.

8.20

(a) In homework 5, question 3.23, we showed that every interval contains rational and irrational numbers.
Therefore, for any $a < b$ in **R**, we have an interval [a, b] and there is some rational number q in it. Therefore, **Q** is dense in **R**.

(b) By question 3.23, there exists some irrational number c in the interval [a, b]. Therefore, **R\Q** is dense in **R.**

Homework 11

2. (i) $C = \bigcup_{n\geq 1} (-\infty, r_n) \cap \mathbb{Q}$ is a Dedekind cut

$r_1 \leq r_2 \leq r_3 \leq \cdots$

Obviously, $C \subset \mathbb{Q}$. Also, since $M \in \mathbb{Q}$ and $M \notin C$,

C is a proper subset of $\mathbb{Q}$

C has no maximal element because if C has a maximal element C_k,

then $C_k < r_{t_1}$ for some t_1, but for $t_2 > t_1$, $r_{t_2} > r_{t_1}$

so ~~we for $C_{k2} \in (-\infty, r_{t2}$~~

for some $C_{k2} \in (r_{t_1}, r_{t_2}) \cap \mathbb{Q}$; $C_{k2} \in C$ and $C_{k2} > C_k$.

A contradiction. Therefore, C has no maximal element.

Now, assume $a, b \in \mathbb{Q}$, $a \in C$ and $b < a$.

Since $a \in C$, $a \in (-\infty, r_a) \cap \mathbb{Q}$ for some r_a.

since $b < a$ and $b \in \mathbb{Q}$, $b \in (-\infty, r_a) \cap \mathbb{Q}$.

Thus, $b \in C$. To conclude, C is a Dedekind cut.

(ii) (iii) WLOG, let the infinite decimal be in $[0,1]$ and its $0.\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5\ldots$

~~$S_1 = \frac{a_1}{10}, S_2 = \frac{a_2}{100}, S_3 = \frac{a_3}{1000} \cdots, S_1 \leq S_2 \leq S_3 \leq \cdots$~~

(ii) let $r_1 = \frac{\bar{a}_1}{10}$, $r_2 = \frac{\bar{a}_1}{10} + \frac{\bar{a}_2}{100}$, $\ldots r_n = \sum_{i=1}^{n} \frac{\bar{a}_i}{10^i}$

Then, the Dedekind cut corresponds to the decimal $0.a_1a_2a_3a_4\ldots\ldots$

(iii) Let $a_n = \sum_{i=1}^{n} \bar{a}_i \cdot 10^{n-i}$ Then, $S_n = \frac{a_n}{10^n} = \sum_{i=1}^{n} \bar{a}_i \cdot 10^{-i}$

$\{S_n\}$ has the property that $S_1 \leq S_2 \leq S_3 \ldots\ldots$

Obviously, $S_n = r_n$ so $C = \bigcup_{n\geq 1} [(-\infty, r_n) \cap \mathbb{Q}]$

$= \bigcup_{n\geq 1} [(-\infty, S_n) \cap \mathbb{Q}]$

If a is a rational number less than r_n,

we can have $S_n = r_n > a$. ✠

3. When $x \neq 0$

$$\frac{x^n + \sum_{i<n} a_i x^i}{x^n} = 1 + \sum_{i<n} a_i x^{i-n}$$

$$\lim_{x \to \infty} 1 + \sum_{i<n} a_i x^{i-n} = 1 > 0$$

so $\lim_{x \to \infty} x^n (1 + \sum_{i<n} a_i x^{i-n}) > 0$

$\exists\, k_1 \in \mathbb{R}$, $k_1 > 0$ s.t. $k_1^n (1 + \sum_{i<n} a_i k_1^{i-n})$

$= k_1^n + \sum_{i<n} a_i k_1^i = M$ for some $M > 0$.

~~Since n is odd, we have~~ $\cancel{\lim_{n \to \infty} x^n = \infty}$

Similarly, $\lim_{x \to -\infty} \frac{x^n (1 + \sum_{i<n} a_i x^{i-n})}{x^n} = \lim_{x \to -\infty} 1 + \sum_{i<n} a_i x^{i-n} = 1$

$\lim_{x \to -\infty} x^n < 0 \quad \Rightarrow \lim_{x \to -\infty} x^n (1 + \sum_{i<n} a_i x^{i-n}) < 0$

$\Rightarrow \exists\, k_2 \in \mathbb{R}$, $k_2 < 0$ s.t. $k_2^n (1 + \sum_{i<n} a_i k_2^{i-n}) = M_2$ for $M_2 < 0$

By Intermediate Value Theorem, we can conclude that $\exists\, k_3 \in (k_2, k_1)$ s.t. $k_3^n + \sum_{i<n} a_i k_3^i = 0$

$\Rightarrow x^n + \sum_{i<n} a_i x^i$ with $a_i \in \mathbb{R}$ and n odd has at least one root