§1. F totally real field

$\mathcal{O}_F$

$d = [F:\mathbb{Q}]$, fix $F \hookrightarrow \mathbb{R}^d$.

$\mathfrak{n}$ integral ideal of $\mathcal{O}_F$

k positive integer

Want: Hilbert modular forms over F of wt k and level $\mathfrak{n}$

$GL_2^+(\mathbb{R}) = \{ g \in GL_2(\mathbb{R}) \mid \det g > 0 \} \curvearrowright \mathcal{H}$

$GL_2^+(\mathbb{R})^d \curvearrowright \mathcal{H}^d$

Let $GL_2^+(F) = \{ g \in GL_2(F) \mid \det g \gg 0 \} \curvearrowright \mathcal{H}^d$

For $\mathfrak{b}$ fraction ideal, $\mathfrak{c}$ integral ideal of $\mathcal{O}_F$

Let $\Gamma = \Gamma(\mathfrak{b}, \mathfrak{c}) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(F) \;\middle|\; \begin{matrix} a, d \in \mathcal{O}_F^\times, \ ad - bc \in \mathcal{O}_F^\times \\ b \in \mathfrak{b}^{-1}, \ c \in \mathfrak{b}\mathfrak{c} \end{matrix} \right\}$

ψ_0 charater of $(\mathcal{O}_F/\mathfrak{c})^\times$

Define Hilbert modular form on Γ of wt k and charater ψ_0 to be

$$f : \mathcal{H}^d \to \mathbb{C} \quad \text{holomorphic}$$

s.t. $f\|_k \alpha = \psi_0(\alpha) f$ for every $\alpha \in \Gamma$

where $f\|_k \alpha(z) = (\det \alpha)^{k/2} (cz + d)^{-k} f(\alpha(z))$

$$(\det \alpha)^{k/2} = \left(\prod \det(\alpha_i) \right)^{k/2}$$

$$(cz+d)^{-k} = \prod (c_i z_i + d_i)^{-k}$$

$$\psi_0(\alpha) = \psi_0(a \bmod \mathfrak{c})$$

(if $F = \mathbb{Q}$ also require f holomorphic at ∞)

if $[F:\mathbb{Q}] > 1$ f auto. holo. at ∞ by Koecher's principle

The space of all such forms $M_k(\Gamma, \psi_0)$.

Let δ difference of $F/\mathbb{Q}$

h strict class numbers of F

$t_1, \dots, t_h$ representatives for strict ideal classes

of F, chosen s.t. $N t_\lambda$ prime to $N(\mathfrak{n}p)$

Then let $M_k(\mathfrak{n}, \psi_0) = \prod_{\lambda=1}^{h} M_k(\Gamma(t_\lambda \delta, \mathfrak{n}), \psi_0)$

and $M_k(\mathfrak{n}) = \bigoplus_{\psi_0} M_k(\mathfrak{n}, \psi_0)$

ψ_0: character of $(O_F/\mathfrak{n})^\times$.

1.2

Adelic point of view.

Fix $\mathfrak{n}$ integral ideal of $\mathcal{O}_F$

For $\mathfrak{p}$ prime ideal of $\mathcal{O}_F$

let $Y_\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\mathfrak{p}) \;\middle|\; \begin{array}{l} a\mathcal{O}_{F,\mathfrak{p}} + \mathfrak{n}_\mathfrak{p} = \mathcal{O}_{F,\mathfrak{p}} \\ b \in \delta_\mathfrak{p}^{-1} \\ c \in \mathfrak{n}_\mathfrak{p}\delta_\mathfrak{p} \\ d \in \mathcal{O}_{F,\mathfrak{p}} \end{array} \right\}$

$W_\mathfrak{p} = \{ x \in Y_\mathfrak{p} \mid \det x \in \mathcal{O}_{F,\mathfrak{p}}^\times \}$

$G = \mathrm{Res}_{F/\mathbb{Q}} GL_2$, $G_\mathbb{A} = GL_2(\mathbb{A}_F)$, $G_\mathbb{Q} = GL_2(F)$

$G_{\infty,+} = GL_2^+(\mathbb{R})^d$

$Y = G_\mathbb{A} \cap \left(G_{\infty,+} \times \prod_\mathfrak{p} Y_\mathfrak{p} \right)$

$W = G_{\infty,+} \times \prod_\mathfrak{p} W_\mathfrak{p}$

put $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix}$, $x_\lambda^{-1} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow G_{\mathbb{A}} = \coprod_{\lambda=1}^{h} G_{\mathbb{Q}} x_\lambda W = \coprod_{\lambda=1}^{h} G_{\mathbb{Q}} x_\lambda^{-1} W$

Put $\Gamma_\lambda = x_\lambda W x_\lambda^{-1} \cap G_{\mathbb{Q}} = \Gamma(t_\lambda \delta, n)$

ψ_0 character of $(\mathcal{O}_F/n)^\times$

gives rise to $\psi_\Gamma : \Gamma \to \mathbb{C}^\times$

by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \psi_0(a_n \bmod n)$

Then $M_k(\Gamma(t_\lambda \delta, n), \psi_0)$

$= \{ f \cdots \mid f\|_k \gamma = \psi_\Gamma(x_\lambda^{-1} \gamma x_\lambda) f,\ \gamma \in \Gamma_\lambda \}$

Let $(f_1, \cdots, f_h) \in M_k(n, \psi_0)$

define $\mathbb{C}$-valued function $\mathbf{f}$ on $G_{\mathbb{A}}$ by

$\alpha \in G_{\mathbb{Q}}, w \in W$

$\phi(\alpha x_\lambda^{-\iota} w) = \psi_\Upsilon(w^\iota)\,(f_\lambda \|_k w_\infty)(i, i, \dots, i)$

ϕ also determines $(f_1, \dots, f_n)$

$\phi(\beta x v) = \psi_\Upsilon(v^\iota)\,\phi(x)$, $\beta \in G_{\mathbb{Q}}$, $v \in W$ with $v_\infty = 1$.

Hecke action

- $y \in \Upsilon$, let $WyW = \coprod_j W y_j$ with $(y_j)_\infty = 1$

$f \in M_k(\mathfrak{n}, \psi_0)$

define $(\phi | WyW)(x) = \sum_j \psi_\Upsilon(y_j)^{-1}\,\phi(x y_j^\iota)$, $x \in G_{\mathbb{A}}$

- $\alpha \in G_{\mathbb{Q}} \cap x_\lambda \Upsilon x_\mu^{-1}$

$$\Gamma_\lambda \alpha \Gamma_\mu = \coprod_j \Gamma_\lambda \alpha_j$$

$f \in M_k(\Gamma_\lambda, \psi_0) \rightsquigarrow f|_{\Gamma_\lambda \alpha \Gamma_\mu} \in M_k(\Gamma_\mu, \psi_0)$

$$f|_{\Gamma_\lambda \alpha \Gamma_\mu} = \sum_j \psi_\gamma (x_\lambda^{-1} \alpha_j x_\mu)^{-1} f||_{\alpha_j}$$

- $y \in W$, for each λ, we can find

$\alpha_\lambda \in x_\lambda Y x_\mu^{-1} \cap G_{\mathbb{Q}}$ s.t. (for a unique determined μ)

$W y W = W x_\lambda^{-1} \alpha_\lambda x_\mu W$

then $f|WyW = (g_1, \dots, g_n)$

$$g_\mu = f_\lambda | \Gamma_\lambda \alpha_\lambda \Gamma_\mu .$$

m integral ideal of F

$T(m) = \sum_y WyW$, $y \in Y$ s.t. $\det y \cdot O_F = m$

$S(m) = WaW$, $a \in A_F^\times$ s.t. $m = a \cdot O_F$

if m prime to n

$= 0$ if m not prime to n

$R(W, T)$ generated by $T(p), S(p)$ for all p prime

ideal is a commutative alg.

and $T(m)\,T(n) = \sum_{m+n \subset a} N(a) S(a) T(a^{-2}mn)$

Fourier expansion.

For $f_\lambda \in M_k(\Gamma(t_\lambda \delta, n), \psi_0)$

it is inv. under $\begin{pmatrix} 1 & (t_\lambda\delta)^{-1} \\ 0 & 1 \end{pmatrix}$

hence can be represented by a Fourier expansion

$$f_\lambda = a_\lambda(0) + \sum_{\substack{\mu \in t_\lambda \\ \mu \gg 0}} a_\lambda(\mu)\, e^{2\pi i\, tr(\mu z)}$$

$\mu \gg 0$ holomorphy condition

$\mu \in t_\lambda$, $b \in (t_\lambda \delta)^{-1}$, $\mu b \in O_F O_F^\vee$

$\Rightarrow tr(\mu b) \in \mathbb{Z}$.

$tr(\mu z) = \mu_1 z_1 + \cdots + \mu_d z_d$.

Such f_λ is called a cusp form if $a_\lambda(0)$ is zero in the expansion of each $f\|_k \alpha$ with $\alpha \in GL_2^+(F)$.

We get spaces of cusp forms

$$S_k(n), \quad S_k(n, \psi_0).$$

Every non-zero integral ideal $\mathfrak{a}$ of O_F can be written as $(\mu) \cdot t_\lambda^{-1}$ for some λ and $\mu \in t_\lambda$, $\mu \gg 0$.

Let $f \in M_k(n)$.

Define $c(\mathfrak{a}, f) = \begin{cases} a_\lambda(\mu)\, N(\mu)^{-k/2} & \text{if } \mathfrak{a} = (\mu)\cdot t_\lambda^{-1} \text{ integral} \\ \text{independent of } \mu. & \\ 0 & \text{otherwise} \end{cases}$

also write $c_\lambda(0, f) = a_\lambda(0)\cdot N(t_\lambda)^{-k/2}$

$\lambda = 1, \cdots, h.$

The Dirichlet series

$D(s, f) = \sum c(\mathfrak{a}, f)\, N\mathfrak{a}^{-s}.$ converges for $\mathrm{Re}\, s \gg 0$

continued to meromorphic function, entire if f cusp form

Shimura: $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, $f \in M_k(\mathfrak{n}, \psi_0)$

then $\exists\ f^\sigma \in M_k(\mathfrak{n}, \psi_0^\sigma)$

s.t. $c(\mathfrak{a}, f^\sigma) = c(\mathfrak{a}, f)^\sigma$

$\forall\ \mathfrak{a}, \lambda$

$c_\lambda(0, f^\sigma) = c_\lambda(0, f)^\sigma$

For $A \subset \mathbb{C}$ we say $f \in M_k(\mathfrak{n}, A)$

if $c(\mathfrak{a}, f),\ c_\lambda(0, f) \in A$ $\forall\ \mathfrak{a}, \lambda$

Shimura: $M_k(\mathfrak{n}, \mathbb{Z}) \otimes \mathbb{C} = M_k(\mathfrak{n}, \mathbb{C})$

Fix $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$, $\bar{\mathbb{Q}} \subset \mathbb{C}$.

For A contained in $\bar{\mathbb{Q}}_p$ or $\mathbb{C}$

$$M_k(n, A) = M_k(n, \mathbb{Z}) \otimes A$$

is the same as $f \in M_k(n, \mathbb{C})$

s.t. $c(\mathfrak{a}, f)$, $c_\lambda(0, f) \in A$.

More generally if A is a field containing $\mathbb{Q}$ or $\mathbb{Q}_p$, let $M_k(n, A) = M_k(n, \mathbb{Z}) \otimes A$ and for subring R of A containing $\mathbb{Z}$ or $\mathbb{Z}_p$,

$$M_k(n, R) = \{ f \in M_k(n, A) \mid c(\mathfrak{a}, f), c_\lambda(0, f) \in R \}.$$

There is a theory of Hecke operators on $M_k(n)$ given by

$\{ T_n(\mathfrak{a}), S_n(\mathfrak{a}) \}_{\mathfrak{a} \text{ integral ideal}}$

§2.

p prime, $\mathcal{O}$ finite extension of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$

$\Lambda = \mathcal{O}[\![T]\!]$

$\mathbb{Q}_\infty$ cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$

Let $p^e = [F \cap \mathbb{Q}_\infty : \mathbb{Q}]$

if p odd, let $q = p$

$p = 2$ $\quad q = 4$

$u = (1+q)^{p^e}$

For each $k \in \mathbb{Z}$, $\zeta \in \mu_{p^\infty}$

define $v_{k,\zeta} : \Lambda \longrightarrow \mathcal{O}[\zeta]$

$$v_{k,\zeta}(1+T) = \zeta u^{k-2}$$

Let $\mathfrak{X} = \{(k, \zeta) \mid k \geq 2, \ \zeta^{p^r} = 1$ for some $r \geq 0\}$

Def. A Λ-adic modular form $\mathcal{F}$ over F of level $\mathfrak{n}$ is a set of elements of Λ given by

$$\begin{cases} c(\mathfrak{a}, \mathcal{F}) & \mathfrak{a} \text{ integral ideal of } \mathcal{O}_F \\ c_\lambda(0, \mathcal{F}) & \lambda = 1, \dots, h \end{cases}$$

with the property that for all but finitely many $v = v_{k,\zeta}$ with $(k, \zeta) \in \mathfrak{X}$, there exists

$$f_v \in M_k(\mathfrak{n}p^r, \mathcal{O}[\zeta]) \quad \text{with}$$

Dirichlet series $D(s, f_v) = \sum v(c(\mathfrak{a}, \mathcal{F})) N\mathfrak{a}^{-s}$

and constant terms of f_v are given by

$$c_\lambda(0, f_v) = v(c_\lambda(0, \mathcal{F})).$$

Such f_v is called a specialization

For Λ-adic form $\mathcal{F}$

let $A_{\mathcal{F}} = \{ (k, \zeta) \mid k \geq 2,\ \zeta^{p^r} = 1,\ r \geq 0,$

$f_v \in M_k(Np^r)\quad v = v_{k,\zeta} \}$

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if for each $(k, \zeta) \in A_{\mathcal{F}}$

$$(Na)^{k-2} S(a)\, v(\mathcal{F}) = v(\lambda_a)\, v(\mathcal{F})$$

for some $\lambda_a \in \Lambda$

then define $S(a)^* \mathcal{F} = \lambda_a \mathcal{F}$.

Iff assume for each a, λ_a exists then

$\exists$ Λ-adic form g s.t. $v g = T(a) v \mathcal{F}$

for each $v = v_{k,\zeta}$, $(k, \zeta) \in A_{\mathcal{F}}$

define $T(a) \mathcal{F} = g$.

We can define $S(\mathfrak{a})$, $T(\mathfrak{a})$ action on a

Explicitly Λ-adic form compatible with specializations

$$c(m, T(n)\mathcal{F}) = \sum_{m+n\subset\mathfrak{a}} \lambda_{\mathfrak{a}} \cdot N\mathfrak{a} \cdot c(\mathfrak{a}^{-2}mn, \mathcal{F})$$

$$c_{\lambda}(0, T(\mathfrak{a})\mathcal{F}) = \sum_{\mathfrak{b}|\mathfrak{a}} \lambda_{\mathfrak{b}} \cdot N\mathfrak{b} \cdot c_{\lambda\rho^2\mu^{-1}}(0, \mathcal{F})$$

where $[\mathfrak{b}] = [t_\rho]$, $[\mathfrak{a}] = [t_\mu]$.

suppose $S(\mathfrak{a})\cdot\mathcal{F} = \lambda_{\mathfrak{a}}\mathcal{F}$

Suppose now the and correspondence $\mathfrak{a} \longmapsto \lambda_{\mathfrak{a}}$ comes from a character $\chi : \varprojlim I_{np^r} \longrightarrow \Lambda = \mathcal{O}[[T]]$
cont. hom

where $I_{\mathfrak{f}}$ denote the strict ray class group of F mod $\mathfrak{f}$ s.t. $\chi(\mathfrak{a}) = \lambda_{\mathfrak{a}}$ for all $\mathfrak{a}$ prime to np. ($S(\mathfrak{a}) = 0$ if $\mathfrak{a}$ not prime to np^r)

We then say $\mathcal{F}$ has charater χ and denote the Λ-module $\mathcal{M}_\Lambda(n, \chi) = \left\{ \begin{matrix} \mathcal{F} \text{ of level } n \\ \text{has char } \chi \end{matrix} \right\}$

$$\cup\!\!\mid$$

$$S(n, \chi) = \left\{ \begin{matrix} \mathcal{F} \in \mathcal{M}(n, \chi) \\ f_\nu \text{ always cusp} \end{matrix} \right\}$$

next ordinary projector.

Write $U_p = T_{np}(p)$

and let $e = \lim_r U_p^{r!}$ acting on $M_k(np^r, \mathcal{O})$

$r \geq 1, k \geq 1$

with topology $\|f\| = \sup_{\mathfrak{a}, \lambda} \{ \|c(\mathfrak{a}, f)\|_p, \|C_\lambda(0, f)\|_p \}$

Claim: e is defined.

Let $\mathfrak{P}$ maximal ideal of $\mathcal{O}$, $\mathbb{F} = \mathcal{O}/\mathfrak{P}$.

$X = M_k(np^r, \mathcal{O}) / \mathfrak{P} M_k(np^r, \mathcal{O})$ $\mathbb{F}$-v.s.

U_p acts on X can be s+n

semisimple, nilpotent

$\exists\ p^{\mu}$ large enough, $U_p^{p^{\mu}}$ semisimple

$\Rightarrow \exists\ \beta$, U_p^{β} idempotent on X.

$\Rightarrow$ $U_p^{p^n \beta}$ idempotent of $M_\kappa / \mathfrak{P}^n M_\kappa$. //

$e^2 = e$.

Prop. e extends naturally to $\mathcal{M}(n, X)$ compatible with specialization and $e^2 = e$.

Pf. $g \in \mathcal{M}$, $A_g = \bigcup_{i=1}^{\infty} A_i$, $A_i \subset A_{i+1}$ finite s.t.

$(\kappa, \zeta) \in A_i \Rightarrow (\kappa, \zeta^\sigma) \in A_i$, $\sigma \in Gal(\bar{\mathbb{Q}}_p / K)$, K: Frac $\mathcal{O}$

$\mathcal{M}(A_i) \subset \mathcal{M}_g \subset \mathcal{M}$

$$\mathcal{M}_g / \mathcal{M}(A_i) \hookrightarrow \bigoplus_{\{(\kappa,\zeta)\} \in A_i} M_\kappa(np^r, \mathcal{O}[\zeta])$$

$\Rightarrow$ $e\mathfrak{g} = \hat{\mathfrak{g}_i}$ mod $\mathcal{M}(A_i)$

compatible and take limit $\hat{\mathfrak{g}} \in \mathcal{M}$. ///

Write $\mathcal{M}^\circ = e\mathcal{M}$, $M^\circ_K = eM_K$.

For ideal $\mathfrak{a}$ of O_F, let $F_\mathfrak{a}$ be the strict ray class field of conductor $\mathfrak{a}\cdot\infty$

By CFT, define group $\mathcal{E}_r$ s.t. via Artin map

$$\mathrm{Gal}(F_n(\zeta_{p^r})/F) \simeq I_{np^r}/\mathcal{E}_r.$$

Def. χ is called cyclotomic if $\forall\ v = v_{\zeta}$, ζ of order exactly p^r

(i) $\chi_v(\mathcal{E}_r) = 1$

(ii) χ_v has order $\geq cp^r$ $c > 0$ independent of v

Thm. χ cyclotomic $\Rightarrow$ $M^\circ(n, \chi)$ f.g. Λ-mod.

lemma. For n, $\dim S_k^\circ(np, k)$ is independent of k.

$K = \mathrm{Frac}\, \mathcal{O}$.

lemma. $\Rightarrow$ Thm.

Pick $g_1, \ldots, g_t \in M^\circ$ independent over Λ.

Let $N = \langle g_1, \ldots, g_t \rangle_\Lambda \simeq \Lambda^t \subset M^\circ$. ($M$ torsion-free over Λ)

Consider $N / (1+T-u^{k-2})N \longrightarrow M_k^\circ(np, \mathcal{O})$

$g \longmapsto g(u^{k-2}-1)$

For $k \gg 0$ it is inj. and $\mathcal{O}$-rank of the RHS is bounded by lemma and trivial bound on Eis.

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§3. Extend classical Eisenstein series to Λ-adic.

For χ strict ideal class char. of conductor n

let $\mathcal{O}_\chi = \mathbb{Z}_p[\chi]$.

Define $\tilde{\chi} : \varprojlim I_{np^r} \longrightarrow \mathcal{O}_\chi [\![T]\!]$

$$a \longmapsto \chi(a)(1+T)^a$$

if a prime to np

where $N a = u^a \delta$, $\delta \in \mu_{p-1}$, $a \in \mathbb{Z}_p$.

Then $\tilde{\chi}$ is cyclotomic.

Classical Eisenstein series.

Fix integer $k \geq 1$

For a strict ray class character ε of conductor f and parity $(-1)^k$ at each ∞-prime there is a modular form $E_{k,\varepsilon} \in M_k(f, \varepsilon)$

whose Dirichlet series is given by

$$\zeta_F(s) L_f(s+1-k, \varepsilon)$$

and constant terms are

$$C_\lambda(0, E_{k,\varepsilon}) = 2^{-d} L(1-k, \varepsilon).$$

Prop. For each even strict ray class character χ

of conductor n, $\exists$ $\mathcal{E}_\chi \in M(np, \tilde{\chi}) \otimes F_\Lambda$

$(F_\Lambda = \mathrm{Frac}\,\Lambda,\ \Lambda = \mathbb{Z}_p[\chi][[T]])$.

s.t. $\mathcal{E}_\chi(\zeta u^{k-2} - 1) = E_{k,\chi\rho\omega^{2-k}}$

for all $k \geq 1$, $\zeta \in \mu_{p^\infty}$.

where $\rho = \rho_\zeta : a \longmapsto \zeta^a$ and ω Teichmuller char. of conductor $\mathfrak{g}$, $\omega(\mathfrak{a}) \equiv N\mathfrak{a} \mod \mathfrak{g}$.

Pf. define $\mathcal{E}_\chi$ directly by setting

$$\sum c(m, \mathcal{E}_\chi) N m^{-s} = \zeta_F(s) L(s-1, \tilde{\chi})$$

and $c_\lambda(0, \mathcal{E}_\chi) = 2^{-d} G_\chi(T) / H_\chi(T)$.

Verify the specialization property by Deligne-Ribet p-adic L-function.

$L_p(s, \chi)$ interpolates classical L-values

and satisfy $L_p(1-s, \chi) = G_\chi(u^s-1) / H_\chi(u^s-1)$.

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Next want to extend cusp newforms.

~~For f, g modular forms over F~~

~~define $D(s, f, g) = \sum c(m, f) c(m, g) N m^{-s}$~~

Define $\bar{f}$ s.t. $c(m, \bar{f}) = \overline{c(m, f)}$

Let M_n be the measure of a fundamental domain of $\mathcal{H}^d / \Gamma(t_\lambda \delta, n)$ w.r.t.

$$d\mu(z) = \prod_{v=1}^{d} y_v^{-2} \, dx_v \, dy_v$$

For $f, g \in M_k(n)$ whose product is cusp form

$$\langle f, g \rangle = \sum_{\lambda=1}^{h} M_n^{-1} \int_{\mathcal{H}^d / \Gamma(t_\lambda \delta, n)} \bar{f}(z) g(z) y^k \, d\mu(z)$$

newforms

Prop. For every integral ideal $\mathfrak{a}$, $f \in M_k(\mathfrak{n}, \psi)$

$\exists!$ $f|\mathfrak{a} \in M_k(\mathfrak{a}\mathfrak{n}, \psi)$ s.t.

$$c(\mathfrak{m}, f|\mathfrak{a}) = c(\mathfrak{a}^{-1}\mathfrak{m}, f)$$

Let $S_k'(\mathfrak{n}, \psi) \subset S_k(\mathfrak{n}, \psi)$ spanned by $g|\mathfrak{a}$

for $g \in S_k(\mathfrak{b}, \psi)$, $\mathfrak{b}$ divisors of $\mathfrak{n}$, $\mathfrak{b} \neq \mathfrak{n}$,

$\mathfrak{a}$ runs over all divisors of $\mathfrak{b}^{-1}\mathfrak{n}$

Let $S_k^{\perp}(\mathfrak{n}, \psi) = S_k'(\mathfrak{n}, \psi)^{\perp}$.

This is stable under $T(\mathfrak{m})$ for all $\mathfrak{m}$.

Define a newform to be an element in $S_k^{\perp}(\mathfrak{n}, \psi)$

normalized ($c(\mathcal{O}_F, f) = 1$) eigenform for all $T(\mathfrak{m})$

A newform f (with coeff. in $\mathcal{O}$) is called ordinary if $c(p, f)$ is a unit in $\mathcal{O}$. If f has level n, then $ef^{\neq 0}$ is an eigenform of level $n \cdot \wp$ where

$$\wp = \prod_{\substack{p \mid \wp \\ p \nmid n}} p$$

and is called the p-stabilized newform associated with f.

§4.

In last section, we extend classical Eis to Λ-adic Eis naturally so that classical Eis occurs as specializations of Λ-adic ones.

This section : do the same for cuspidal newforms.

Fix $\overline{F}_{\mathbb{Z}_p[[T]]}$ of $F_{\mathbb{Z}_p[[T]]}$ containing $\overline{\mathbb{Q}}_p$ and assume all extensions L considered below is in $\overline{F}_{\mathbb{Z}_p[[T]]}$.

Note that $S_\Lambda^\circ(\bar{n}, \tilde{\chi}) \overset{\text{def}}{=} \bigcup_{r=0}^{\infty} S_\Lambda^\circ(np^r, \tilde{\chi})$

f.g. over Λ.

For L/F_Λ finite extension (we may choose $\Lambda = \mathbb{Z}_p[X][[T]]$)

let $S_L^\circ(n, \tilde{\chi}) = S_{F_\Lambda}^\circ(n, \tilde{\chi}) \otimes L$

Let O_L integral closure of Λ in L

$S_{O_L}^\circ(\bar{n}, \tilde{\chi})$ submod. whose defining data are in O_L.

Fix ζ of order p^r, $O[\zeta] \subset O_r$.

$K_r = \mathrm{Frac}\, O_r$, $\Lambda_r = O_r[[T]]$.

$S_{\Lambda_r}^\circ = S_{\Lambda_r}^\circ(\bar{n}, \tilde{\chi})$, $M_{\Lambda_r}^\circ = M_{\Lambda_r}^\circ(\bar{n}, \tilde{\chi})$.

$$\varphi_{k,\zeta}: M_{\Lambda_r}^\circ / (1+T-\zeta u^{k-2}) M_{\Lambda_r}^\circ \longrightarrow M_k(np^r, O_r, \chi p \omega^{2-k})$$

well-defined and injective for almost all k.

Let n_0 greatest divisor of n prime to p.

Suppose f_k is p-stab. newform of wt k

occurring in the standard decomp. for some

element in $\operatorname{Im} \phi_{k,3}$

($\forall$ cusp form f of level n,

$\exists !$ $f = \sum \beta_i f_i(g_i z)$

$\beta_i \in \mathbb{C}$, f_i normalized newform of

level m_i, $m_i g_i \mid n$)

Then if n_0 divides level of f_k

$\Rightarrow$ no $f_k(gz)$ occurs in such a

decomp. for $g \neq (1)$

choose a combination $t = \sum_{i=1}^{s} a_i T(n_i)$, $a_i \in \mathcal{O}'$

K'/K_T finite ext. s.t.

$$t : M_k^\circ(Np^r, K', \chi p \omega^{2-k}) \to K'\{f_k\}$$

Let $t f_k = c f_k$, $c \neq 0$.

t acts on f.g. Λ'-mod $m^\circ_{\Lambda_r} \otimes \Lambda'$, $\Lambda' = \mathcal{O}'[[T]]$

has char. poly. $g(x) \in \Lambda'[x]$ as t integral over Λ'.

Let $\Upsilon = 1 + T - 3u^{k-2}$

$\Rightarrow$ $g(x) \bmod \Upsilon$ = char.poly $(t | m^\circ_{\Lambda_r} \otimes \Lambda' / \Upsilon \cdots)$

= char.poly $(t | m^\circ_{\Lambda_r} / \Upsilon \otimes \mathcal{O}')$

$\varphi_{k,3} \Rightarrow$ $g(x) \equiv x^{\mu-1}(x - c) \bmod \Upsilon$

$\mu = \deg g(x)$.

Let L splitting field of $g(x)$ over F_n

P extension of Υ to O_L

$\Rightarrow \exists!$ root α of $g(x)$ in O_L s.t.

$\alpha \equiv c \mod P$

Let $h(x) = g(x) / x - \alpha \in O_L[x]$

$\Rightarrow h(t) M_L^{\circ}$ is 1-dim.

By hypothesis $\exists\, g \in M_{\Lambda_r}^{\circ}$ s.t. f_K occurs in

$\phi_{K,3}(g)$. Then $h(t) g$ is eigenfunction for all

Hecke operators and mod P gives non-zero multiple

of f_K

Thm. Suppose f_k is a p-stabilized newform in $S_k^0(\mathfrak{a}, \alpha)$ with coeff. in $\mathcal{O}$, a finite extension of $\mathbb{Z}_p$ contained in $\overline{\mathbb{Q}}_p$.

Then there exists an eigenform $g \in S^0_{\mathcal{O}_L}(\bar{\mathfrak{a}}, \alpha\omega^{k-2})$ for some finite extension L of F_n s.t.

$$g \equiv f_k \mod \tilde{P}_k \quad \text{for some } \tilde{P}_k \text{ of } \mathcal{O}_L$$

extending the prime $P_k = ((1+T) - u^{k-2})$ of Λ.

Pf. Choose primitive even char. χ s.t.

(i) $\chi\alpha^{-1}$ is char. of $\mathrm{Gal}(F(\zeta_{p^\infty})/F)$

(ii) $\chi\omega^{3-k}$ has conductor divisible by p

(iii) $\mathrm{cond}(\alpha^{-1}\chi\omega^{3-k}) \mid \mathrm{cond}(\chi\omega^{3-k})$

Let $\mathfrak{c} = \mathrm{cond}(\chi)$, set $\mathfrak{m} = \mathfrak{a}/(\mathfrak{c}, \mathfrak{a})$.

Pick $t \geq 1$, let $g_{k,t} = g_{\mathfrak{m}\mathfrak{p}^r, \chi, \psi}(u^{k-2} - 1)$

(if χ primitive even strict ray class char. of cond. $\mathfrak{c}$, we have $\mathcal{E}_\chi \in \mathcal{M}(\mathfrak{c}\mathfrak{p}, \tilde{\chi})$, write $\mathfrak{n} = \mathfrak{m}\mathfrak{p}^r \cdot \mathfrak{c}$

define $\mathcal{E}_{\mathfrak{m}\mathfrak{p}^r, \chi} = \sum_{\mathfrak{b} \mid \mathfrak{m}\mathfrak{p}^r} \mu(\mathfrak{b}) \tilde{\chi}^{-1}(\mathfrak{b}) N\mathfrak{b}^{-2} \mathcal{E}_\chi(\mathfrak{m}\mathfrak{p}^r \cdot \mathfrak{b}^{-1} \cdot z)$

$\Rightarrow \mathcal{E}_{\mathfrak{m}\mathfrak{p}^r, \chi} \in \mathcal{M}(\mathfrak{n}, \hat{\chi})$

For $\mathfrak{d}$ integral ideal, $h \in M_1(\mathfrak{d}, \psi)$ set

$$g_{\mathfrak{m}\mathfrak{p}^r, \chi, \psi} = h\, \mathcal{E}_{\mathfrak{m}\mathfrak{p}^r, \chi}(u^{-1}(1+T) - 1) \in \mathcal{M}(\mathrm{l.c.m.}(\mathfrak{n}, \mathfrak{d}), \tilde{\chi}\psi\omega))$$

We want to choose g s.t.

$$\langle f_k, e g_{k,t} \rangle \neq 0$$

This can be done by choosing h carefully.

$e g_{mp^r, \chi, \psi}(T) \in M^\circ_\Lambda(\bar{a}, \alpha w^{k-2})$

$\Lambda = \mathcal{O}[[T]], \quad \mathcal{O} = \mathcal{O}[\chi, \psi].$

and $(f_k, e g_{mp^r, \chi, \psi}(u^{k-2} - 1)) \neq 0$

$\Rightarrow$ we may lift f_k for k large enough

to a normalized eigenform $\mathcal{F} \in S^\circ_{\mathcal{O}_L}(\bar{a}, \alpha w^{k-2})$

Then we can also lift eigenforms

$f - c(g, f) f(gz)$ lifts to $\mathcal{F} - c(g, \mathcal{F}) \mathcal{F}(gz)$

$\dim_L S^\circ_L(\bar{a}, \alpha w^{k-2})$ bounded independent of L

$\Rightarrow$ there are at most finitely many

distinct normalized Λ-adic eigenforms

$\mathcal{F}_1, \ldots, \mathcal{F}_t$ of level $\bar{a}$ and char. $\alpha \omega^{k-2}$

Let L be a finite extension of F_λ containing $\mathcal{O}$ over which all $\mathcal{F}_i$ is defined

There are stabilized eigenform $g_j^{(m)}, j=1, \ldots, t_m$ of wt $k' = k + 2^i p^m (p-1)$ s.t.

$$g_j^{(m)} \equiv f_k \mod \mathfrak{P}_m^{x_j(m)}, \quad p^{m/2} \mid \mathfrak{P}_m^{\sum x_j(m)}$$

$\mathfrak{P}_m$ maximal ideal of $\mathcal{O}_{\mathfrak{P}_m}$, $\mathcal{O}_{\mathfrak{P}_m}$ composite of $\mathcal{O}$ and integral closure of $\mathcal{O}_L / \tilde{P}_k$.

As $m \to \infty$, $\exists$ infinite sequence of $g_j^{(m)}$ s.t.

$$v_p(\mathfrak{P}_m^{x_j(m)}) \to 0$$

and $\exists$ infinite subsequence S s.t.

$g_j^{(m)}$ is a specialization of g_i

$\Rightarrow g_i \equiv g_j^{(m)} \mod \tilde{P}_{K'}$, $\quad g_j^{(m)} \equiv f_K \mod \mathfrak{P}_m^{x_j(m)}$

Let $I \subset O_L$,

$I = \{ x \in O_L : x \mod \tilde{P}_{K'} \equiv 0 \mod \mathfrak{P}_m^{x_j(m)} \quad \forall m \in S \}$

suppose $x \in I \cap \Lambda$

if $m \in S$, $m \geq N$, $\mathfrak{P}_m^{x_j(m)} \equiv 0 \mod p^N$

$x \in \Lambda / P_{K'} \hookrightarrow O_L / \tilde{P}_{K'} \hookrightarrow O_{G_m}$

is divisible by p^N

$\Rightarrow x \in (P_{K'}, p^N) = (P_K, p^N)$

since $\Lambda (P_K, p^N) = P_K \Rightarrow I \cap \Lambda \subset P_K$

$\Rightarrow I$ contained in some $\tilde{P}_K$ above P_K

and $g_i \equiv f_k \mod I$.