Seminar Notes : Diamonds I

0. Motivations.

For a perfectoid space X, the functor $X \longmapsto X^\flat$ has many good properties.

What about analytic adic spaces over $\mathbb{Z}_p$?

There should be a functor $\{\text{analytic adic spaces over } \mathbb{Z}_p\} \longrightarrow ?$

$$X \longmapsto X^\diamond$$

which forgets the structure morphism to $\mathbb{Z}_p$ but keeps all topological information.

Fact : $X/\mathbb{Z}_p$ analytic adic space, then X is pro-étale locally perfectoid

$$X = \mathrm{coeq}(\tilde{X} \times_X \tilde{X} \rightrightarrows \tilde{X})$$

where $\tilde{X} \to X$ pro-étale covering, $\tilde{X}$, $R = \tilde{X} \times_X \tilde{X}$ perfectoid.

Then $X^\diamond$ should be $\mathrm{coeq}(R^\flat \rightrightarrows \tilde{X}^\flat)$.

What category does this live in ?

1. Complements on Perfectoid Spaces.

Perfd : the category of perfectoid spaces

Perf : the subcategory of perfd spaces in char p

Recall : A perfd space X is an adic space covered by opens of the form $\mathrm{Spa}(R, R^+)$ where R is perfd ring.

$X = \mathrm{Spa}(R, R^+)$ perfd is called affinoid perfd (affd perfd) if R perfd.

Inverse limit.

Prop. Let $X_i = \mathrm{Spa}(R_i, R_i^+)$ be a cofiltered inverse system of affinoid perfd spaces. For any $\varpi \in R_i^+$ compatible choice of pseudouniformizers for large i, let

R^+ be the ϖ-adic completion of $\varinjlim R_i^+$ and $R = R^+[\frac{1}{\varpi}]$.

Then $X = \mathrm{Spa}(R, R^+)$ is again affinoid perfd. Moreover

(0) The map $|X| \to \varprojlim |X_i|$ is a homeomorphism of spectral spaces.

(i) The base change functor $(X_i)_{fet} \to X_{fet}$ induces equivalence of categories

$\{(X_i)_{fet}\}_{i \in I}$ $\quad \mathrm{Hom}(Y_i, Y_j)$

$$2\text{-}\varinjlim (X_i)_{fet} \longrightarrow X_{fet}$$

$2\text{-}\varinjlim (X_i)_{fet}$ $\quad = \varinjlim_{\substack{i \to k \\ j \to k}} \mathrm{Hom}_{X_k}(Y_i^k, Y_j^k)$

(ii) Same as in (i) with fet replaced by each of the following

affinoid etale $\subset$ qc sep. etale $\subset$ qcqs etale.

$Y \to X$ etale, Y affinoid.

Pf. Sketch.

(0) follows from $\mathrm{Spa}(\hat{R}, \hat{R}^+) \xrightarrow{\sim} \mathrm{Spa}(R, R^+)$ is homeomorphism of spectral spaces.

(i) follows from the theorem of Gabber-Ramero. ///

(A, A^+) Tate pair, $\varpi \in A^+$ s.t. A Henselian along ϖA^+

$\Rightarrow A_{fet} \simeq \hat{A}_{fet}$, $\hat{A}$ ϖ-adic completion.

Pro-etale Morphisms

Def. A morphism $f: \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfd is called affinoid pro-etale if there is a filtered direct system (A_i, A_i^+) with A_i perfd and $\mathrm{Spa}(A_i, A_i^+) \xrightarrow{f_i} \mathrm{Spa}(A, A^+)$ etale such that $(B, B^+) = \widehat{\varinjlim (A_i, A_i^+)}$ and f is induced by f_i.

$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A_i, A_i^+) \xrightarrow{f_i} \mathrm{Spa}(A, A^+)$

A morphism $f: X \to Y$ of perfd spaces is called pro-etale if it is locally on the source and target affinoid pro-etale, i.e. $\forall x \in X$, $\exists x \in U \subset X$ open, $V \subset Y$ open, $f(U) \subset V$ and $f|_U : U \to V$ is affinoid pro-etale.

Examples.

1. Profinite sets.

X perfd space, S profinite set.

Let $X \times \underline{S}$ be the inverse limit of $X \times S_i$ where $S = \varprojlim S_i$, S_i finite set.

Then $X \times \underline{S}$ is perfd and $X \times \underline{S} \to X$ is pro-etale.

$X \sqcup X \to X$ etale.

$X = \mathrm{Spa}(R, R^+)$

$X \sqcup X = \mathrm{Spa}(R \times R, \dots)$

2. $X = \mathrm{Spa}\, \mathbb{Q}_p$.

$\tilde{X} = \mathrm{Spa}\, \mathbb{Q}_p^{cycl} = \mathrm{Spa}\, \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$ is a perfd field and $\tilde{X} \to X$ pro-etale cover.

$\mathbb{Q}_p \to \mathbb{Q}_p(\mu_{p^n})$ $\quad$ $\mathbb{Q}_p(\mu_{p^\infty})$

3. Inclusion of points.

K perfd field, S profinite set, $s \in S$.

$Y = \mathrm{Spa}(K, \mathcal{O}_K) \times \underline{S}$

$f : X = \mathrm{Spa}(K, \mathcal{O}_K) \times \{s\} \longrightarrow Y$ is proetale since $X = \varprojlim \mathrm{Spa}(K, \mathcal{O}_K) \times \underline{U_i}$

where $U_i \subset S$ open and closed and $\bigcap_i U_i = \{s\}$.

Actually any Zariski closed immersions are pro-etale.

$X = \mathrm{Spa}(R, R^+)$ $\qquad$ $R \to R/I = S$

$Z = \mathrm{Spa}(S, S^+)$

Lemma.

(i) Let $g: Z \to Y$, $f: Y \to X$ be pro-etale / affnd pro-etale morphisms then $fg: Z \to X$ is pro-etale / affnd pro-etale.

(ii) Let $f: Y \to X$ be pro-etale / affnd pro-etale, $g: X' \to X$ be a map from perfd / affnd perfd space. Then $f': X' \times_X Y \to X'$ is pro-etale / affnd pro-etale.

(iii) Let
$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[dr, bend right, "g"'] & Y' \arrow[d, "g'"] \\ & X \end{tikzcd}$$
be a commutative diagram with g, g' pro-etale / affnd pro-etale then h is pro-etale / affnd pro-etale.

Pf.

(i) Locally assume g, f affnd pro-etale.

Write $Z = \varprojlim Z_i \xrightarrow{g} Y$, $Z_i \xrightarrow{g_i} Y$ g_i affnd etale, $Y = \varprojlim Y_j$

but $2\text{-}\varinjlim (Y_j)_{\text{affnd, et}} = (Y)_{\text{affnd, et}}$ so $\exists$ $Z_{ij} \longrightarrow Y_j$ affnd etale

such that $Z_i = Z_{ij} \times_{Y_j} Y$ and $Z_i = \varprojlim_j Z_{ij}$.

Then $Z_{ij} \to Y_j \to X$ affnd etale and $Z = \varprojlim_i \varprojlim_j Z_{ij} \xrightarrow{h} X$ pro-etale.

(ii) Assume all affnd and check directly.

(iii) Assume all affnd pro-etale.

$$\begin{tikzcd} & Y \times_X Y' \arrow[d, "\tilde{g}"] \\ Y \arrow[ur, "s"] \arrow[r, "h"] \arrow[d, "g"'] & Y' \arrow[dl, "g'"] \\ X & \end{tikzcd}$$

$\tilde{g}$ is base change of g hence affnd pro-etale.

Enough to show sections of affnd pro-etale is affnd pro-etale. Suppose $Y = \varprojlim Y_i \to X$ and $s: X \to Y$ is a section, then there are sections $s_i: X \to Y_i$ hence $X = \varprojlim (X \times_{Y_i} Y) \to Y$ is pro-etale. //

Prop. Let X be affnd perfd.

X^{aff}_{et} be the category of etale maps $f: Y \to X$ from affnd perfd Y

$X^{aff}_{pro\text{-}et}$ ... affnd pro-etale ...

Then the functor $\mathrm{Pro}(X^{aff}_{et}) \longrightarrow X^{aff}_{pro\text{-}et}$, $\{Y_i\} \longmapsto$ "$\varprojlim$" Y_i

is an equivalence of categories.

Pf. By definition the functor is essentially surjective. Enough to show fully faithful.

Write $X = \mathrm{Spa}(R, R^+)$, $Y_i = \mathrm{Spa}(S_i, S_i^+)$, $Z_j = \mathrm{Spa}(T_j, T_j^+)$

$Y = \mathrm{Spa}(S, S^+)$, $Z = \mathrm{Spa}(T, T^+)$

Need to show $\mathrm{Hom}_X(Y, Z) = \underbrace{\varprojlim_j \varinjlim_i \mathrm{Hom}_X(Y_i, Z_j)}$.

This is the $\mathrm{Mor}(\{Y_i\}, \{Z_j\})$ in Pro category.

WLOG assume J is a singleton so $Z \to X$ etale. Need to show

$\mathrm{Hom}_X(\varprojlim Y_i, Z) = \mathrm{Hom}_X(Y, Z) = \varinjlim_i \mathrm{Hom}_X(Y_i, Z)$.

This can be checked locally on $Z = \mathrm{Spa}(T, T^+)$.

Since locally an etale map is a composition of rational embedding and finite etale map

WLOG assume $f: Z \to X$ is one of these.

① f rational embedding: $U = \mathrm{Spa}(T, T^+) \hookrightarrow X = \mathrm{Spa}(R, R^+)$.

For any $g \in \mathrm{Hom}_X(Y, U)$, $Y = \varprojlim Y_i$

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr] \arrow[ddr] & & U \arrow[ddl] \\ & Y_i \arrow[ur] \arrow[d] & \\ & X & \end{tikzcd}$$

actually since $U \hookrightarrow X$ open immersion, such g is unique if exists.

Then the map $\varprojlim Y_i \longrightarrow X$ factors through U.

Compactness argument.

$$\begin{tikzcd} \varprojlim Y_i \arrow[r] \arrow[d] & Y_{i'} \arrow[dl] \arrow[d] \\ U \arrow[r, hook] & X \end{tikzcd}$$

Topologically we have $|Y| = \varprojlim |Y_i|$ thus

$$|Y| \setminus \{\text{premage of } U\} = \varprojlim |Y_i| \setminus \{\text{premage of } U\}$$

For each i, $|Y_i| \setminus \{\text{premage of } U\}$ is closed in a spectral space hence spectral, and it is compact and Hausdorff for the constructible topology. Then the inverse limit is empty implies one of them has to be empty. (Tychonoff's Theorem).

Hence the map $Y \longrightarrow U \hookrightarrow X$ factors through some Y_i:

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & U \arrow[r, hook] & X \\ & Y_i \arrow[ur] & & \end{tikzcd}$$

② f is finite etale.

Recall we have equivalence of categories

$$\{\text{finite etale } S_\infty\text{-algebras}\} = \underline{2\text{-}\varinjlim}\ \{\text{finite etale } S_i\text{-algebras}\}$$

Hence $\mathrm{Hom}_X(Y, Z) = \mathrm{Hom}_Y(Y, Y\times_X Z) = \underline{\varinjlim \mathrm{Hom}_{Y_i}(Y_i, Y_i \times_X Z)} = \varinjlim \mathrm{Hom}_X(Y_i, Z)$

///

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "id"'] & Y\times_X Z \arrow[dl] \\ Y & \end{tikzcd}$$

$Y = \varprojlim Y_i$

$Y\times_X Z = \varprojlim Y_i \times_X Z$

$$\begin{tikzcd} Z\times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

$Z\times_X Y \to Y$ finite etale

$Z \to X$ finite etale

$$\varinjlim_{\substack{i\to k \\ j \to k}} \mathrm{Hom}_{Y_k}(Y_i \times_{Y_k} Y_k,\ (Y_j \times_X Z)\times_{Y_j} Y_k)$$

Def. (The big pro-étale site)

Consider the following categories

- Perfd : perfectoid spaces
- Perf : perfectoid spaces in char p
- $X_{\text{pro-et}}$: perfd spaces pro-étale over X

Endow them the structure of a site by saying that a collection $\{f_i : Y_i \to Y\}_I$ is a (pro-étale) covering if

- f_i pro-étale, $\forall i \in I$.
- $\forall$ qc open $U \subset Y$, $\exists$ finite subset $I_U \subset I$, qc open $U_i \subset Y_i$, $\forall i \in I_U$ such that $U = \bigcup_{i \in I_U} f_i(U_i)$.

fpqc

Prop. Let X be a perfd space.

1. The presheaves $\mathcal{O}_X, \mathcal{O}_X^+$ are sheaves on Perfd. If X affinoid then for $i > 0$

$$H^i(X_{\text{pro-et}}, \mathcal{O}_X) = 0 \quad , \quad H^i(X_{\text{pro-et}}, \mathcal{O}_X^+) \text{ almost zero}.$$

2. The presheaf on Perfd defined by h_X is a sheaf, i.e. all representable presheaves are sheaves, i.e. pro-étale site is subcanonical.

exact exact

$Y = \cup U$, U affinoid perfd.

$$\{f_i : Y_i \to Y\}$$

$$\begin{tikzcd}
 & \mathcal{O}_Y \arrow[r] \arrow[d] & \prod \mathcal{O}_{Y_i} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod \mathcal{O}_{Y_i \times_Y Y_j} \arrow[d] & \text{equalizer} \\
\text{exact} \arrow[r] & \prod \mathcal{O}_U \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \cdots \arrow[d, shift left] \arrow[d, shift right] & \\
 & \prod \mathcal{O}_{U \cap U_i} \arrow[r] & \cdots \arrow[r, shift left] \arrow[r, shift right] & \cdots &
\end{tikzcd}$$

Pf.

1. WLOG assume $X = \mathrm{Spa}(R, R^+)$ affinoid perfd since they are already sheaves for the analytic topology.

We only need to check the sheaf property and almost vanishing of Čech cohomology relative to affinoid pro-étale covering $Y = \mathrm{Spa}(R_\infty, R_\infty^+) \longrightarrow X$.

Fix a pseudouniformizer $\varpi \in R$. preshaf: $\mathcal{O}_X^+/\varpi$

Claim the Čech complex $0 \to R^+/\varpi \longrightarrow R_\infty^+/\varpi \to \cdots$ is almost exact.

For each j, $0 \to R^+/\varpi \to R_j^+/\varpi \to \cdots$ is almost exact. R_j etale over R

Take direct limit over j, a filtered direct limit of almost exact sequences is almost exact.

$0 \to R^+ \to R_\infty^+ \to \cdots$

$\downarrow \varpi$

Thus $0 \to R^+ \to R_\infty^+ \to \cdots$ is almost exact as all terms are ϖ-torsion free

$\downarrow$

and ϖ-adically complete. Hence higher cohomology of $\mathcal{O}_X^+$ almost vanish.

$0 \to R^+/\varpi$

Invert ϖ, $0 \to R \to R_\infty \to \cdots$ is exact hence $\mathcal{O}_X$ is pro-etale sheaf whose higher cohomology is zero.

Then $\mathcal{O}_X^+$ is also a sheaf as it is determined by subsheaf of $\mathcal{O}_X$ of functions bounded by 1 everywhere. h_X is pro-etale sheaf.

2. Reduce to the case $X = \mathrm{Spa}(R, R^+)$, $Y = \mathrm{Spa}(S, S^+)$ affinid.

Let $\{f_i : Y_i \to Y\}$ be a pro-etale cover on which one has compatible maps $Y_i \to X$. Then we get $R \to H^0(Y_{proet}, \mathcal{O}_Y) = S$ and similarly $R^+ \to S^+$. Hence there is a map $Y \to X$. ///

$Y_i \times_Y Y_j \rightrightarrows X$

2. Definition of Diamonds.

X analytic adic space $/\mathbb{Z}_p$

$R = \tilde{X} \times_X \tilde{X} \rightrightarrows \tilde{X} \to X$ $\qquad$ $X^{\diamond} = \mathrm{coeq}(R^{\flat} \rightrightarrows \tilde{X}^{\flat})$

perf.

Intuitively $D = \tilde{X}^{\flat} / R^{\flat}$ where $\tilde{X}^{\flat}$, $R^{\flat}$ are perf spaces.

Def. A diamond is a pro-étale sheaf D on Perf such that one can write $D = X/R$ as a quotient by a perf space X and an equivalence relation $R \subset X \times X$ such that R is a perfd space with s.t. $R \to X$ pro-étale.

Question: What kind of pair (R, X) gives rise to a diamond X/R ?

We need R injects into $X \times X$.

Injection morphisms.

Def. A map $f: Y \to X$ of perfd spaces is called injection if for all Z perfd space the map $f_* : \mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$ is injective.

A map $f: Y \to X$ of perfd spaces is called an immersion if it is an injection and $|f| : |Y| \to |X|$ is a locally closed immersion. If $|f|$ is in addition open / closed then f is called an open / closed immersion.

$S \overset{f}{\underset{g}{\rightrightarrows}} R \hookrightarrow X \times X$ open immersion

$f = g$

Prop. Let $f : Y \to X$ in perfd. TFAE.

1. f injection, i.e. $\forall\, T$, $\mathrm{Hom}(T, Y) \hookrightarrow \mathrm{Hom}(T, X)$

2. $\forall$ affnd perfd fields (K, K^+), $\mathrm{Hom}(\mathrm{Spa}(K, K^+), Y) \hookrightarrow \mathrm{Hom}(\mathrm{Spa}(K, K^+), X)$

3. $\forall$ affnd alg. closed fields (C, C^+). ...

4. The map $|f| : |Y| \to |X|$ is injective and $\forall\, y \in Y$, $K(y) \xrightarrow{\sim} K(f(y))$.

5. The map $|f| : |Y| \to |X|$ is injective and $\forall\, T$.

$$\mathrm{Hom}(T, Y) \xrightarrow{\sim} \mathrm{Hom}(T, X) \times_{C^0(|T|, |X|)} C^0(|T|, |Y|)$$

is bijection where C^0 denotes the set of continuous functions.

$$\begin{tikzcd} & Y \arrow[r] & X \\ T \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd} \qquad \begin{tikzcd} & {|Y|} \arrow[r, hook, "\text{inj.}"] & {|X|} \\ {|T|} \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

Prop 5.3 Et on Diamonds

$X, Y \in \mathrm{Perf}$ $\quad Z \in \mathrm{Perf}$ $\quad Z \to X$, $Z \to Y$

s.t. $\forall\ T \to X$, $T \to Y$ $\quad \exists!\ T \to Z \begin{smallmatrix} \to X \\ \to Y \end{smallmatrix}$

Prop. Let $R \overset{s}{\underset{t}{\rightrightarrows}} X$ in Perf . s.t pro-étale such that the induced map $R \to X \times X$ is an injection making R an equivalence relation. Then $D = X/R$ is a diamond and the natural map $R \longrightarrow X \times_D X$ is isomorphism of sheaves.

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, hook] & X \times_D X \arrow[d, hook] \\ & X \times X \end{tikzcd}$$

Pf. $D = X/R$ is a diamond by definition.

To show $R \xrightarrow{\sim} X \times_D X$, note that both are subsheaves of $X \times X$ so $\underline{R \hookrightarrow X \times_D X}$.

$h_R(Z) \to h_{X \times_D X}(Z)$

For any Z perf with $Z \longrightarrow X \times_D X$, there are two maps $a, b : Z \longrightarrow X$ such that the induced maps $Z \to D$ agree. This implies there is some pro-étale covering $\tilde{Z} \to Z$ such that $\tilde{Z} \to Z \to X \times X$ factors through R. Then the map $\tilde{Z} \to R$ could be descended to $Z \to R$ since we have

$$\begin{tikzcd} \tilde{Z} \times_Z \tilde{Z} \arrow[r, "p_1"] \arrow[d, "p_2"'] & \tilde{Z} \arrow[d] \\ \tilde{Z} \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} \tilde{Z} \times_Z \tilde{Z} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[drr, bend right] & \tilde{Z} \arrow[r] & R \arrow[r, hook] & X \times X \\ & & Z \arrow[ur, bend right] & \end{tikzcd}$$

$\Rightarrow \tilde{Z} \times_Z \tilde{Z} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \tilde{Z} \longrightarrow R$ agree and h_R is a sheaf.

Then we get a map $Z \longrightarrow R$ and the composition $Z \longrightarrow R \hookrightarrow X \times X$ is just $Z \xrightarrow{(a,b)} X \times X$ since after pro-étale covering $\tilde{Z}$ this is true.

Hence $R \to X \times_D X$ is surjective. $\qquad h_{X \times X}$ $\qquad$ ///

Lemma. The absolute product of two perf spaces is again perf.

Pf. Suffice to show the product of two affnd perf space is again affnd perf.

Let $X = \mathrm{Spa}(A, A^+)$, $Y = \mathrm{Spa}(B, B^+)$, pseudouniformizer $\varpi \in A$, $\varpi' \in B$.

For each $m, n \geq 1$, define a topology on $A \otimes_{\mathbb{F}_p} B$ using ring of definition

$$(A^\circ \otimes_{\mathbb{F}_p} B^\circ)\left[\frac{\varpi^m \otimes 1}{1 \otimes \varpi'}, \frac{1 \otimes (\varpi')^n}{\varpi \otimes 1}\right]$$ equipped with $\varpi \otimes 1$-adic topology.
(same as $1 \otimes \varpi'$-adic topology)

Let $(A \otimes_{\mathbb{F}_p} B)^+$ be the integral closure of $A^+ \otimes_{\mathbb{F}_p} B^+$ in $A \otimes_{\mathbb{F}_p} B$.

Then $(A \otimes_{\mathbb{F}_p} B, (A \otimes_{\mathbb{F}_p} B)^+)$ is a Huber pair and let $(C_{m,n}, C_{m,n}^+)$ be its completion. Then $C_{m,n}$ is a perfect Tate ring hence perfd.

For $m' \geq m$ and $n' \geq n$ there is a natural open immersion

$$\mathrm{Spa}(C_{m,n}, C_{m,n}^+) \hookrightarrow \mathrm{Spa}(C_{m',n'}, C_{m',n'}^+).$$

Claim: the union of these over all pairs (m,n) represents the product.

Suppose (R, R^+) is a complete Huber pair

$$f : (A, A^+) \to (R, R^+), \quad g : (B, B^+) \to (R, R^+)$$

Then the sequence $f(\varpi)^m / g(\varpi')$, $g(\varpi')^n / f(\varpi)$ both approach 0 in R

there exists $m, n \geq 1$ such that both lie in A°.

Then the homomorphism $(A \otimes_{\mathbb{F}_p} B, A^+ \otimes_{\mathbb{F}_p} B^+) \to (R, R^+)$ factors uniquely

through some $(C_{m,n}, C_{m,n}^+) \to (R, R^+)$. ///

Prop. Let D, D' be diamonds. Then the product sheaf $D \times D'$ is a diamond.

Pf. Write $D = X/R$, $D' = X'/R'$ where X, R, X', R' are perf spaces and $R \hookrightarrow X \times X$, $R' \longrightarrow X' \times X'$ are injections inducing equivalence relations.

Then $X \times X'$, $R \times R'$ are perf spaces and $R \times R' \longrightarrow (X \times X') \times (X \times X')$ is an injection inducing equivalence relation. Therefore $D \times D' = X \times X' / R \times R'$ is a diamond. fiber products of diamonds are diamond. ///

Example.

$X = \mathrm{Spa}\, \mathbb{Q}_p$ has a pro-étale perfd covering $\tilde{X} = \mathrm{Spa}\, \mathbb{Q}_p^{cycl}$.

$\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) = \mathbb{Z}_p^\times$. $(\mathbb{Q}_p^{cycl})^\flat = \mathbb{F}_p((t^{\frac{1}{p^\infty}}))$ t-adic completion of $\mathbb{F}_p(t^{\frac{1}{p^\infty}})$

where $t = (1, \zeta_p, \zeta_{p^2}, \cdots) - 1$ for $\{\zeta_{p^n}\}$ compatible system of p-th power roots of unity.

$\mathbb{Z}_p^\times$ acts on $(\mathbb{Q}_p^{cycl})^\flat$ by $\gamma \cdot t = (1+t)^\gamma - 1$. $\gamma \in \mathbb{Z}_p^\times$

$R = \tilde{X} \times_X \tilde{X} = \tilde{X} \times \underline{\mathbb{Z}_p^\times}$ considered as a perfd space.

$\mathrm{Spd}\, \mathbb{Q}_p = X^\diamond = \mathrm{coeq}(\tilde{X}^\flat \times \underline{\mathbb{Z}_p^\times} \rightrightarrows \tilde{X}^\flat) = \mathrm{Spa}\, \mathbb{F}_p((t^{\frac{1}{p^\infty}})) / \mathbb{Z}_p^\times$

Prop. $X = \mathrm{Spa}(R, R^+)$ affinoid perf space, then $(\mathrm{Spd}\, \mathbb{Q}_p)(X)$ is the set of isom. classes of the following data $(R \to \tilde{R}, t \in \tilde{R})$ where

1. $R \to \tilde{R}$ is a $\mathbb{Z}_p^\times$-torsor, i.e. $\tilde{R} = (\varinjlim R_n)^\wedge$. R_n/R finite étale with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ $R = \mathbb{Q}_p$, $R_n = \mathbb{Q}_p(\mu_{p^n})$

2. $t \in \tilde{R}$ topologically nilpotent unit such that $\forall \gamma \in \mathbb{Z}_p^\times$, $\gamma \cdot t = (1+t)^\gamma - 1$.

Introduction

1. Motivation from global Langlands on function fields

$X/\mathbb{F}_p$ sm. proj. curve, $K(X)$ function field. Drinfeld & L. Lafforgue

$$\left\{ \text{cuspidal auto. rep. of } GL_n(\mathbb{A}_K) \right\} \xrightarrow[\text{(roughly 1:1)}]{\pi \longmapsto \sigma(\pi)} \left\{ \text{irr. rep. } \mathrm{Gal}_K \to GL_n(\overline{\mathbb{Q}}_\ell) \right\}$$

Key innovation : Drinfeld Shtuka

Def. A shtuka of rank n over $S/\mathbb{F}_p$ is $(\mathcal{E}, \varphi_\mathcal{E})$.

$\mathcal{E} \in \mathrm{Vect}_n(S \times_{\mathbb{F}_p} X)$, $\varphi_\mathcal{E} : \mathrm{Frob}_S^* \mathcal{E} \dashrightarrow \mathcal{E}$ mero. $\simeq$ defined on open $U \subset S \times_{\mathbb{F}_p} X$ fibrewise dense in X, $\mathrm{Frob}_S : S \times_{\mathbb{F}_p} X \xrightarrow{(\mathrm{Frob}_p, \mathrm{id}_X)} S \times_{\mathbb{F}_p} X$.

Moduli spaces of shtukas $\Rightarrow$ Langlands correspondence

$(\mathcal{E}, \varphi_\mathcal{E})$ rank n over $S = \mathrm{Spec}\, k$, k alg. closed. Consider

- $x_1, \dots, x_m \in X(k)$ where $\varphi_\mathcal{E}$ undefined : legs of shtuka
- $\mu_1, \dots, \mu_m$ cocharacters $\mathbb{G}_m \to GL_n$ encoding $\varphi_\mathcal{E}$ near x_i

$x \in X(k)$ a leg, $t \in \mathcal{O}_{X,x}$ uniformizer

$(\mathrm{Frob}_S^* \mathcal{E})^\wedge_x$ and $\mathcal{E}^\wedge_x$ are free rank n modules over

$$\mathcal{O}^\wedge_{X_k, x} \simeq k[\![t]\!]$$