Algebraic Stacks and Artin's Axioms

Contents

1 Prestacks and stacks

1.1 Sites

We would like to form topologies on a scheme finer than Zariski topology.

Definition 1.1. A Grothendieck topology on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$ there is a set $\mathrm{Cov}(X)$ each of whose elements is a collection of morphisms $\{X_i \to X\}_I$ in $\mathcal{S}$ satisfying

(1) (identity) If $X' \to X$ is an isomorphism then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_I \in \mathrm{Cov}(X)$ and $Y \to X$ is any morphism then for each i the fibre product $X_i \times_X Y$ exists and $\{X_i \times_X Y \to Y\}_I \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_I \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$ then $\{X_{ij} \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$.

A site is a category with a Grothendieck topology.

Example 1.1. If X is a scheme, the big Zariski (resp. etale) site is the category Sch/X where a covering of a scheme U over X is a collection of open immersions (resp. etale morphisms) $\{U_i \to U\}$ such that $\coprod \mathrm{Im}\, U_i = U$. We denote this site as $(\mathrm{Sch}/X)_{Zar}$ (resp. $(\mathrm{Sch}/X)_{et}$).

1.2 Prestacks

Let $\mathcal{S}$ be a category and $p\colon \mathfrak{X} \to \mathcal{S}$ be a functor of categories. We visualize this as

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, mapsto] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

where a, b are objects of $\mathfrak{X}$ and S, T are objects of $\mathcal{S}$. We say a is over S and α is over f if $p(a) = S$ and $p(\alpha) = f$.

Definition 1.2. A functor $p\colon \mathfrak{X} \to \mathcal{S}$ is a prestack over $\mathcal{S}$ if

(1) (pullback exist) for any diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto, dashed] & b \arrow[d, mapsto] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

of solid arrows, there exist a morphism $a \to b$ over $S \to T$;

(2) (universal property for pullbacks) for any diagram

$$\begin{tikzcd} a \arrow[r, dashed] \arrow[d, mapsto] \arrow[rr, bend left] & b \arrow[r] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & T \end{tikzcd}$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

Remark 1.2. Axiom 2 implies that the pullback in Axiom 1 is unique up to unique isomorphism. We often write f^*b or $b|_S$ to indicate **a choice** of pullback.

Definition 1.3. If $\mathfrak{X}$ is a prestack over $\mathcal{S}$, the fibre category $\mathfrak{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathfrak{X}$ over $\mathcal{S}$ with morphisms over id_S.

Remark 1.3. The fibre category $\mathfrak{X}(S)$ is a groupoid.

Example 1.4 (Presheaves are prestacks)**.** If $F\colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf, then we can construct a prestack $\mathfrak{X}_F$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$. A map $(a', S') \to (a, S)$ is a map $f\colon S' \to S$ such that $a' = F(f)a$. We often abuse notation by conflating F and $\mathfrak{X}_F$.

For a scheme X, applying the previous construction to the functor $\mathrm{Mor}(-, X)\colon \mathrm{Sch} \to \mathrm{Sets}$ yields a prestack $\mathfrak{X}_X$ and we will just refer to $\mathfrak{X}_X$ as X.

Example 1.5 (Prestack of smooth curves)**.** Define the prestack $\mathcal{M}_g$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$ of genus g, i.e. smooth and proper morphism $\mathcal{C} \to S$ such that every geometric fibre is a connected curve of genus g. A map $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is a pair $(\alpha\colon \mathcal{C}' \to \mathcal{C}, f\colon S' \to S)$ such that the diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "\alpha"] \arrow[d] & \mathcal{C} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is Cartesian. The fibre category $\mathcal{M}_g(\mathbb{C})$ is the groupoid of smooth connected projective complex curves C of genus g such that $\mathrm{Mor}(C, C') = \mathrm{Isom}_{\mathrm{Sch}/\mathbb{C}}(C, C')$.

Definition 1.4 (Quotient and classifying prestacks)**.** Let $G \to S$ be a group scheme acting on a scheme $U \to S$ via $\sigma\colon G \times_S U \to U$. We define the quotient prestack $[U/G]^{pre}$ as the category over Sch/S where the fibre category over $T \to S$ is the quotient groupoid $[U(T)/G(T)]$ of the group $G(T)$ acting on the set $U(T)$. A morphism $(T' \to U) \to (T \to U)$ over $T' \to T$ is an element $\gamma \in G(T')$ such that $(T' \to U) = \gamma \cdot (T' \to T \to U) \in U(T')$.

Define the prestack $[U/G]$ as the category over Sch/S whose objects over $T \to S$ are diagrams

$$T \longleftarrow P \xrightarrow{f} U$$

where $P \to T$ is a principal G-bundle and $f\colon P \to U$ is a G-equivariant morphism. A morphism

$$(g\colon T' \to T, \varphi\colon P' \to P)\colon (P' \to T', P' \xrightarrow{f'} U) \to (P \to T, P \xrightarrow{f} U)$$

is a pair of morphisms of schemes such that

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[rr, bend left, "f'"] \arrow[d] & P \arrow[r, "f"] \arrow[d] & U \\ T' \arrow[r, "g"] & T & \end{tikzcd}$$

commutes with the left square Cartesian.

Define the classifying prestack as $\mathbf{B}G = [S/G]$ arising as the special case where $U = S$.

1.3 Morphisms of prestacks

Definition 1.5.

(1) A morphism of prestacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ over $\mathcal{S}$ is a functor such that $p_{\mathfrak{X}}(a) = p_{\mathfrak{Y}}(f(a))$ for every object a of $\mathfrak{X}$.

(2) If $f, g\colon \mathfrak{X} \to \mathfrak{Y}$ are morphisms of prestacks, a 2-morphism (or 2-isomorphism) $\alpha\colon f \to g$ is a natural transformation such that for every object a of $\mathfrak{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathfrak{Y}$ is over the identity in $\mathcal{S}$ (which is necessarily an isomorphism). We shall describe the 2-morphism α as

$$\begin{tikzcd} \mathfrak{X} \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & \mathfrak{Y} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

(3) Define the category $\mathrm{Mor}(\mathfrak{X}, \mathfrak{Y})$ whose objects are morphisms of prestacks from $\mathfrak{X}$ to $\mathfrak{Y}$ and whose morphisms are 2-morphisms.

(4) A diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \xrightarrow{\sim} f \circ g'$ is called 2-commutative.

(5) A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of prestacks is an isomorphism (or equivalence) if there exists a morphism $g\colon \mathfrak{Y} \to \mathfrak{X}$ and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathfrak{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathfrak{Y}}$. A prestack $\mathfrak{X}$ is equivalent to a presheaf if

there is a presheaf F and an isomorphism between $\mathfrak{X}$ and $\mathfrak{X}_F$.

Remark 1.6. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of prestacks over $\mathcal{S}$. Then f is fully faithful/isomorphism if and only if $f_S\colon \mathfrak{X}(S) \to \mathfrak{Y}(S)$ is fully faithful/equivalence of categories for every $S \in \mathcal{S}$.

Let $\mathcal{S}$ be a category. For any $S \in \mathcal{S}$, the presheaf $\mathrm{Mor}(-, S)$ can be viewed as a prestack over $\mathcal{S}$ which we will still denote by S.

Lemma 1.7 (The 2-Yoneda Lemma)**.** Let $\mathfrak{X}$ be a prestack over $\mathcal{S}$ and $S \in \mathcal{S}$. Then the functor

$$\mathrm{Mor}(S, \mathfrak{X}) \to \mathfrak{X}(S)\ ,\ f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

We will use the 2-Yoneda lemma to pass between morphisms $S \to \mathfrak{X}$ and objects in $\mathfrak{X}(S)$.

Example 1.8 (Quotient stack presentations)**.** Consider the prestack $[U/G]$ arising from a group action $\sigma\colon G \times_S U \to U$. There is an object of $[U/G]$ over U given by the diagram

$$U \xleftarrow{p_2} G \times_S U \xrightarrow{\sigma} U$$

hence we have a morphism $U \to [U/G]$.

We discuss fibre products for prestacks and give the construction.

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y}' \to \mathfrak{Y}$ be morphisms of prestacks over $\mathcal{S}$. Define the prestack $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ over $\mathcal{S}$ as the category of triples (x, y', γ) where $x \in \mathfrak{X}$, $y' \in \mathfrak{Y}'$ are objects over the same object $p_{\mathfrak{X}}(x) = p_{\mathfrak{Y}'}(y') = S$ in $\mathcal{S}$, and $\gamma\colon f(x) \xrightarrow{\sim} g(y')$ is an isomorphism in $\mathfrak{Y}(S)$. A morphism of $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ consists of a triple (r, ξ, η) where $r\colon p_{\mathfrak{X}}(x_1) = p_{\mathfrak{Y}'}(y_1') \to p_{\mathfrak{X}}(x_2) = p_{\mathfrak{Y}'}(y_2')$ is a morphism in $\mathcal{S}$, $\xi\colon x_1 \xrightarrow{\sim} x_2$ and $\eta\colon y_1' \xrightarrow{\sim} y_2'$ are morphisms in $\mathfrak{X}$ and $\mathfrak{Y}'$ over r such that

$$\begin{tikzcd} f(x_1) \arrow[r, "f(\xi)"] \arrow[d, "\gamma_1"] & f(x_2) \arrow[d, "\gamma_2"] \\ g(y_1') \arrow[r, "g(\eta)"] & g(y_2') \end{tikzcd}$$

commutes.

Let $p_1\colon \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{X}$ and $p_2\colon \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ denote the projections $(x, y', \gamma) \mapsto x$ and $(x, y', \gamma) \mapsto y'$. Then there is a 2-isomorphism $\alpha\colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$ defined by $\alpha_{(x,y',\gamma)}\colon f(x) \xrightarrow{\gamma} g(y')$. This yields a 2-commutative diagram

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\alpha\nearrow"] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

Theorem 1.9. The prestack $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ together with the morphisms p_1 and p_2 and the 2-isomorphism α is final in such diagrams.

Any 2-commutative diagram satisfies the universal property is called a Cartesian diagram.

A very special case of the fibre product is the product. Suppose $\mathfrak{X}$ and $\mathfrak{Y}$ are prestacks over $\mathcal{S}$, then the product $\mathfrak{X} \times \mathfrak{Y}$ is also a prestack over $\mathcal{S}$. For any prestack $\mathfrak{X}$ over $\mathcal{S}$, there is a canonical diagonal morphism $\Delta\colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ which as we shall see actually encodes the stackiness.

Example 1.10. Consider the prestack $[U/G]$ arising from a group action $\sigma\colon G \times_S U \to U$. Then for any object

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

in $[U/G](T)$ the corresponding morphism $T \to [U/G]$ fits into a Cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & U \arrow[d] \\ T \arrow[r] & {[U/G]} \end{tikzcd}$$

where $U \to [U/G]$ is as in Example 1.8.

Example 1.11 (Magic about diagonal)**.** Let $\mathfrak{X}$ be a prestack over $\mathcal{S}$ and $a\colon \mathfrak{Y} \to \mathfrak{X}$, $b\colon \mathfrak{Y}' \to \mathfrak{X}$ be morphisms, then there is a Cartesian diagram

$$\begin{tikzcd} \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathfrak{Y} \times \mathfrak{Y}' \arrow[d, "a\times b"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

Example 1.12 (Isom presheaf)**.** Let $\mathfrak{X}$ be a prestack over $\mathcal{S}$ and a, b be objects over $S \in \mathcal{S}$. Then

$$\begin{aligned} \underline{\mathrm{Isom}}_{\mathfrak{X}(S)}(a,b)\colon \mathcal{S}/S &\longrightarrow \mathrm{Sets} \\ (T \xrightarrow{f} S) &\longmapsto \mathrm{Mor}_{\mathfrak{X}(T)}(f^*a, f^*b) \end{aligned}$$

is a presheaf. There is a Cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathfrak{X}(S)}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & S \arrow[d, "{(a,b)}"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

1.4 Stacks

A stack over a site is a prestack such that objects and morphisms glue uniquely in the Grothendieck topology of the site. Verifying a given prestack is a stack reduces to a descent condition.

Definition 1.6. A prestack $\mathfrak{X}$ over a site $\mathcal{S}$ is a stack if the following conditions hold for any covering $\{S_i \to S\}$ of any $S \in \mathcal{S}$:

(1) (morphisms glue) For objects $a, b \in \mathfrak{X}(S)$ and morphisms $\phi_i\colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$

$$\begin{tikzcd} & a|_{S_i} \arrow[dr] \arrow[drr, bend left, "\phi_i"] & & & & S_i \arrow[dr] & \\ a|_{S_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed, "\exists!\phi"] & b & \text{over} \quad S_{ij} \arrow[ur] \arrow[dr] & & S \\ & a|_{S_j} \arrow[ur] \arrow[urr, bend right, "\phi_j"'] & & & & S_j \arrow[ur] & \end{tikzcd}$$

there exists a unique morphism $\phi\colon a \to b$ with $\phi|_{S_i} = \phi_i$;

(2) (objects glue) For objects a_i over S_i and isomorphisms $\alpha_{ij}\colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ satisfying the cocycle condition

$$\begin{tikzcd} & a_i \arrow[dr, dashed] & & & & S_i \arrow[dr] & \\ a_i|_{S_{ij}} \overset{\alpha_{ij}}{\to} a_j|_{S_{ij}} \arrow[ur] \arrow[dr] & & a & \text{over} & S_{ij} \arrow[ur] \arrow[dr] & & S \\ & a_j \arrow[ur, dashed] & & & & S_j \arrow[ur] & \end{tikzcd}$$

there exists an object a over S and isomorphisms $\phi_i\colon a|_{S_i} \to a_i$ such that $\alpha_{ij} \circ \phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$.

A morphism of stacks is a morphism of prestacks. Fibre products of stacks exist as the fibre product of prestacks.

Example 1.13 (Sheaves and schemes are stacks)**.** If F is a sheaf in the site $\mathcal{S}$ then $\mathfrak{X}_F$ is a stack over the site. Schemes are sheaves on Sch_{et} and schemes give rise to stacks over Sch_{et}.

Example 1.14 (Quotient stacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme $U \to S$. Then the prestack $[U/G]$ is a stack over $(\mathrm{Sch}/S)_{et}$. This follows from etale/fppf descent for morphisms of schemes and G-torsors.

Example 1.15 (Stack of sheaves)**.** Let $\mathrm{Sheaves}$ be the prestack over Sch whose objects are pairs (T, F) where T is a scheme and F is a sheaf on T. A morphism $(T, F) \to (T', F')$ is a pair (f, α) where $f\colon T \to T'$ and $\alpha\colon F' \to f_* F$ is a morphism of sheaves on T' such that the adjoint $f^{-1}F' \to F$ is an isomorphism. Because sheaves and their morphisms glue in the Zariski topology, $\mathrm{Sheaves}$ is a stack over the big Zariski site Sch_{Zar}.

Similarly the prestack QCoh and Bun over Sch_{Zar} parameterizing quasi-coherent sheaves and vector bundles are stacks.

Example 1.16 (Stack of schemes)**.** Define $\mathrm{Schemes}$ as the prestack over Sch whose objects are morphisms $T \to S$ where a morphism $(T \to S) \to (T' \to S')$ is a pair (f, g) where $f\colon T \to T'$ and $g\colon S \to S'$ such that the two compositions $T \to S'$ agree. The projection map takes $T \to S$ to S. Since schemes glue in the Zariski topology, $\mathrm{Schemes}$ is a stack over Sch_{Zar}. However it is not a stack over Sch_{et} as schemes can be glued to algebraic spaces in the etale topology which suggests that there is a stack of algebraic spaces over Sch_{et}.

Proposition 1.17 (Moduli stack of smooth curves)**.** If $g \geq 2$ then $\mathcal{M}_g$ is a stack over Sch_{et}.

Proof. □

To any presheaf there is a sheafification which is left adjoint to the forgetful functor. Similarly there is a stackification functor.

Theorem 1.18 (Stackification)**.** There is functor from prestacks to stacks over any site $\mathcal{S}$ called the stackification $\mathfrak{X} \to \mathfrak{X}^{st}$ such that for any stack $\mathfrak{Y}$ over $\mathcal{S}$ the induced functor

$$\mathrm{Mor}(\mathfrak{X}^{st}, \mathfrak{Y}) \longrightarrow \mathrm{Mor}(\mathfrak{X}, \mathfrak{Y})$$

is an equivalence of categories.

2 Algebraic spaces and stacks

2.1 Definition of Algebraic spaces and stacks

Definition 2.1 (Morphisms representable by schemes)**.** A morphism $\mathfrak{X} \to \mathfrak{Y}$ of prestacks or presheaves over Sch/S is representable by schemes if for every morphism from a scheme $T \to \mathfrak{Y}$ over S, the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} T$ is a scheme.

If $\mathcal{P}$ is a property of morphisms of schemes, e.g. etale or surjective, a morphism $\mathfrak{X} \to \mathfrak{Y}$ of prestacks representable by schemes has property $\mathcal{P}$ if for every morphism from a scheme $T \to \mathfrak{Y}$, the morphism $\mathfrak{X} \times_{\mathfrak{Y}} T \to T$ of schemes has property $\mathcal{P}$.

Definition 2.2. An algebraic space is a sheaf X on Sch_{et} such that there exist a scheme U with a surjective etale morphism $U \to X$ representable by schemes.

The map $U \to X$ is called an etale presentation. Morphisms of algebraic spaces are morphisms as sheaves. Any scheme is an algebraic space.

Definition 2.3 (Representable morphisms)**.** A morphism $\mathfrak{X} \to \mathfrak{Y}$ of prestacks or presheaves over Sch is representable if for every morphism $T \to \mathfrak{Y}$ from a scheme, the fibre product $\mathfrak{X} \times_{\mathfrak{Y}} T$ is an algebraic space.

If $\mathcal{P}$ is a property of morphisms of schemes which is etale-local on the source, e.g. surjective, etale or smooth, we say a representable morphism $\mathfrak{X} \to \mathfrak{Y}$ of prestacks has property $\mathcal{P}$ if for every morphism $T \to \mathfrak{Y}$ from a scheme and etale presentation $U \to \mathfrak{X} \times_{\mathfrak{Y}} T$ by a scheme, the composition $U \to \mathfrak{X} \times_{\mathfrak{Y}} T \to T$ has property $\mathcal{P}$.

Definition 2.4. A Deligne-Mumford stack (resp. algebraic stack) is a stack $\mathfrak{X}$ over Sch_{et} such that there exist a scheme U and a surjective etale (resp. smooth) representable morphism $U \to \mathfrak{X}$.

The morphism $U \to \mathfrak{X}$ is called an etale (resp. smooth) presentation. Morphisms of Deligne-Mumford stacks or algebraic stacks are morphisms as stacks. Any algebraic space is a Deligne-Mumford stack and any scheme, algebraic space or Deligne-Mumford stack is an algebraic stack.

Proposition 2.1. Fibre products exist for algebraic spaces, Deligne-Mumford stacks and algebraic stacks which is the same as the fibre product as stacks (hence also as prestacks).

2.2 Representability of the diagonal

2.3 Properties

Definition 2.5. Let $\mathcal{P}$ be a property of morphisms of schemes.

(1) If $\mathcal{P}$ is stable under composition and base change, etale local (resp. smooth local) on the source and target, then a morphism $\mathfrak{X} \to \mathfrak{Y}$ of Deligne-Mumford (resp. algebraic stacks) has property $\mathcal{P}$ if for all etale (resp. smooth) presentations (equivalently there exist presentations) $V \to \mathfrak{Y}$ and $U \to \mathfrak{X} \times_{\mathfrak{Y}} V$ the composition $U \to V$ has $\mathcal{P}$.

The properties of flatness, smoothness, surjectivity, locally of finite presentation/type are smooth local on the source and target, and the properties of etaleness and unramifiedness are etale local on the source and target.

2.4 Equivalence relations and groupoids

3 Artin's axioms

3.1 Limit preserving

Definition 3.1. Let S be a scheme and $\mathfrak{X}$ be a prestack over (Sch/S). We say $\mathfrak{X}$ is limit preserving if for every affine scheme which is a directed inverse limit of affine schemes $\operatorname{Spec} B = \operatorname{Spec} \operatorname{colim} B_\lambda$ over S we have an equivalence of categories

$$\operatorname{colim} \mathfrak{X}(\operatorname{Spec} B_\lambda) \longrightarrow \mathfrak{X}(\operatorname{Spec} B)$$

which means that

(1) Every object of $\mathfrak{X}(\operatorname{Spec} B)$ is isomorphic to the restriction of an object over $\operatorname{Spec} B_\lambda$ for some λ;

(2) Given objects x, y of $\mathfrak{X}$ over $\operatorname{Spec} B_\lambda$ we have

$$\operatorname{Mor}_{\mathfrak{X}(\operatorname{Spec} B)}(x|_{\operatorname{Spec} B}, y|_{\operatorname{Spec} B}) = \operatorname{colim}_{\lambda' \geq \lambda} \operatorname{Mor}_{\mathfrak{X}(\operatorname{Spec} B_{\lambda'})}(x|_{\operatorname{Spec} B_{\lambda'}}, y|_{\operatorname{Spec} B_{\lambda'}})$$

3.2 Formal objects and versality

Definition 3.2. Let $\mathfrak{X}$ be a prestack over Sch/S for S locally Noetherian scheme.

(1) A formal object $\xi = (R, \xi_n, f_n)$ of $\mathfrak{X}$ consists of a complete local Noetherian S-algebra $(R, \mathfrak{m})$, object ξ_n of $\mathfrak{X}$ over $\operatorname{Spec} R/\mathfrak{m}^n$ and morphisms $f_n \colon \xi_n \to \xi_{n+1}$ of $\mathfrak{X}$ over $\operatorname{Spec} R/\mathfrak{m}^n \to \operatorname{Spec} R/\mathfrak{m}^{n+1}$ such that $R/\mathfrak{m}$ is a field of finite type over S.

(2) A morphism of formal objects $a \colon \xi = (R, \xi_n, f_n) \to \eta = (T, \eta_n, g_n)$ is given by morphisms $a_n \colon \xi_n \to \eta_n$ compatible with f_n and g_n. The morphisms $\{a_n\}$ give rise to compatible morphisms $\operatorname{Spec} R/\mathfrak{m}_R^n \to \operatorname{Spec} T/\mathfrak{m}_T^n$ hence a morphism $a_0 \colon \operatorname{Spec} R \to \operatorname{Spec} T$ over S.

With the above notations, suppose x is an object of $\mathfrak{X}$ over R. We can consider the system of restrictions $\xi_n = x|_{\operatorname{Spec} R/\mathfrak{m}^n}$ endowed with the natural morphisms $\xi_1 \to \xi_2 \to \dots$. Then we get a formal object $\xi = (R, \xi_n)$ of $\mathfrak{X}$. The construction gives a functor

$$\left\{ \begin{array}{c} \text{objects } x \text{ of } \mathcal{X} \text{ over } \operatorname{Spec} R \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S \end{array} \right\} \longrightarrow \left\{ \text{formal objects of } \mathcal{X} \right\}$$

A formal object is called effective if it is in the essential image of the functor.

Definition 3.3. Let $\mathfrak{X}$ be a prestack over S locally Noetherian scheme. For any complete local Noetherian S-algebra $(R, \mathfrak{m})$ with $\operatorname{Spec} R/\mathfrak{m}$ of finite type over S, x the closed point of $\operatorname{Spec} R$, $\hat{\xi}$ object of $\mathfrak{X}$ over $\operatorname{Spec} R$ is called formally versal if

$$\begin{tikzcd}
\operatorname{Spec} \kappa(x) \arrow[r] & \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "\hat{\xi}"] \\
& \operatorname{Spec} B' \arrow[r, "\eta'"] \arrow[ur, dashed] & \mathfrak{X}
\end{tikzcd}$$

for every $B' \to B$ surjection of Artinian S-algebra with residue field $\kappa(x)$, an object η' of $\mathfrak{X}$ over $\operatorname{Spec} B'$ with an isomorphism $\alpha \colon \hat{\xi}|_{\operatorname{Spec} B} \to \eta'|_{\operatorname{Spec} B}$, there is a morphism $\operatorname{Spec} B' \to \operatorname{Spec} R$ fitting in the diagram, an isomorphism $\alpha' \colon \hat{\xi}|_{\operatorname{Spec} B'} \to \eta'$ extending α.

Let $\xi = (R, \xi_n, f_n)$ be a formal object. Then ξ is called formally versal if

$$\begin{tikzcd}
\operatorname{Spec} \kappa(x) \arrow[r] & \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R/\mathfrak{m}^n \arrow[r, dashed] \arrow[dr, "\xi_n"] & \operatorname{Spec} R/\mathfrak{m}^m \arrow[d, "\xi_m"] \\
& \operatorname{Spec} B' \arrow[rru, dashed] \arrow[rr, "\eta'"] & & \mathfrak{X}
\end{tikzcd}$$

for every $B' \to B$ surjection of Artinian S-algebra with residue field $\kappa(x)$, an object η' of $\mathfrak{X}$ over $\operatorname{Spec} B'$ with an isomorphism $\alpha \colon \xi_n|_{\operatorname{Spec} B} \to \eta'|_{\operatorname{Spec} B}$, there is some $m \geq n$ and a morphism $\operatorname{Spec} B' \to \operatorname{Spec} R/\mathfrak{m}^m$ fitting in the diagram, an isomorphism $\alpha' \colon \xi_m|_{\operatorname{Spec} B'} \to \eta'$ extending α.

It is easy to see that a formally versal object $\hat{\xi}$ of $\mathfrak{X}$ over $\operatorname{Spec} R$ gives rise to a formally versal formal object ξ under the functor described above.

3.3 Artin's axioms

For simplicity assume k is an algebraic closed field.

Theorem 3.1 (Artin's axioms)**.** Let $\mathfrak{X}$ be a stack over $(\mathrm{Sch}/k)_{et}$. Then $\mathfrak{X}$ is an algebraic stack locally of finite type over k if and only if the following conditions hold:

(0) (Limit preserving) The stack $\mathfrak{X}$ is limit preserving.

(1) (Representability of the diagonal) The diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable.

(2) (Existence of formal deformations) For every $x \colon \operatorname{Spec} k \to \mathfrak{X}$ there exist a complete local Noetherian k-algebra $(R, \mathfrak{m})$ and a compatible family of morphisms $\xi_n \colon \operatorname{Spec} R/\mathfrak{m}^{n+1} \to \mathfrak{X}$ with $x = \xi_0$ such that $\{\xi_n\}$ is formally versal.

(3) (Effectivity) For every complete local Noetherian k-algebra $(R, \mathfrak{m})$ the natural functor

$$\mathfrak{X}(\operatorname{Spec} R) \longrightarrow \lim \mathfrak{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories.

(4) (Openness of versality) For any morphism $\xi_U : U \to \mathfrak{X}$ where U is a scheme of finite type over k and k-point $u \in U$ such that ξ_U is formally versal at u, then ξ_U is formally versal for all k-points in an open neighborhood of u.

Remark 3.2. In practice, Condition (1) is often easy to verify directly. The representability of diagonal could be translated into the condition that for every scheme T over k and objects $\xi, \eta \colon T \to \mathfrak{X}$ the functor

$$\underline{\mathrm{Isom}}_T(\xi, \eta) \colon \mathrm{Sch}/T \longrightarrow \mathrm{Sets} \ , \ (T' \to T) \longmapsto \mathrm{Mor}_{\mathfrak{X}(T')}(\xi|_{T'}, \eta|_{T'})$$

is representable by an algebraic space.

Condition (2) is often a consequence of Schlessinger and Rim's theorem on existence of formally versal deformations. We shall discuss this in more details.

Condition (3) is often a consequence of Grothendieck's existence theorem.

In some simplified moduli problem, Condition (4) can be checked directly. For example, if for each point $x \colon \operatorname{Spec} k \to \mathfrak{X}$ the formal deformation space $(R, \mathfrak{m})$ as in Condition (2) is regular then Condition

(4) is automatically satisfied. In more general moduli problems, Condition (4) is often guaranteed by a well-behaved deformation and obstruction theory. We shall also discuss this in more details.

Definition 3.4. Let S be a locally Noetherian scheme. Let $\mathfrak{X}$ be a prestack over S. Then we say $\mathfrak{X}$ satisfies the condition (RS) if for every pullback

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

where A, B, C are finite type local Artinian S-algebras and $A \to B$ is surjective, the functor of fibre categories

$$\mathfrak{X}(\operatorname{Spec} A \times_B C) \longrightarrow \mathfrak{X}(\operatorname{Spec} A) \times_{\mathfrak{X}(\operatorname{Spec} B)} \mathfrak{X}(\operatorname{Spec} C)$$

is an equivalence of categories.

Definition 3.5. Let S be a locally Noetherian scheme and $\mathfrak{X}$ is a prestack over S. Let k be a field and $\operatorname{Spec} k \to S$ be a morphism of finite type. Let x_0 be an object of $\mathfrak{X}$ over $\operatorname{Spec} k$. Let $\mathcal{F} = \mathcal{F}_{\mathfrak{X},k,x_0}$ be the category whose

(1) objects are morphisms $x_0 \to x$ of $\mathfrak{X}$ where x is over $\operatorname{Spec} A$ with A an Artinian local ring over S with residue field k;

(2) morphisms $(x_0 \to x) \to (x_0 \to x')$ are commutative diagrams

$$\begin{tikzcd} x & & x' \arrow[ll] \\ & x_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in $\mathfrak{X}$.

The tangent space of $\mathcal{F}$ is defined to be

$$T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{ \begin{matrix} \text{isomorphism classes of morphisms} \\ x_0 \to x \text{ over } \operatorname{Spec}(k) \to \operatorname{Spec}(k[\epsilon]) \end{matrix} \right\}$$

Also we define the infinitesimal automorphism to be

$$\mathsf{Inf}(\mathcal{F}_{\mathcal{X},k,x_0}) = \operatorname{Ker}\left(\mathsf{Aut}_{\operatorname{Spec}(k[\epsilon])}(x_0') \to \mathsf{Aut}_{\operatorname{Spec}(k)}(x_0)\right)$$

where x_0' is the pullback of x_0 to $\operatorname{Spec} k[\epsilon]$.

If $\mathfrak{X}$ satisfies the (RS) condition then $T\mathcal{F}_{\mathfrak{X},k,x_0}$ comes equipped with a natural k-vector space structure and $\mathsf{Inf}(\mathcal{F}_{\mathfrak{X},k,x_0})$ also has a natural k-vector space structure such that addition agrees with composition of automorphisms.

Proposition 3.3. Let S be a locally Noetherian scheme and $\mathfrak{X}$ is a stack over $(\operatorname{Sch}/S)_{et}$. Assume that

(0) The stack $\mathfrak{X}$ is limit preserving.

(1) The diagonal is representable.

(2) The stack $\mathfrak{X}$ satisfies the (RS) condition and the space $T\mathcal{F}_{\mathfrak{X},k,x_0}$ and $\mathsf{Inf}(\mathcal{F}_{\mathfrak{X},k,x_0})$ are finite dimensional for every k and x_0.

(3) Every formal object of $\mathfrak{X}$ is effective.

(4) $\mathfrak{X}$ satisfies openness of versality.

(5) $\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.

Then $\mathfrak{X}$ is an algebraic stack.

Remark 3.4. To deal with openness of versality, we want to use deformation and obstruction theory. To be more precise, we want A-linear functors

$$\mathrm{Aut}_\xi, \mathrm{Def}_\xi \text{ and } \mathrm{Ob}_\xi \colon \text{Finite } A\text{-modules} \to \text{Finite } A\text{-modules}$$

for every finitely generated k-algebra A and object $\xi \colon \operatorname{Spec} A \to \mathfrak{X}$ satisfying extra properties. For many moduli problems, these functors are naturally identified with certain cohomology modules which are easy to verify the extra requirements. For example if $\mathfrak{X} = \mathcal{M}_g$ and ξ corresponds to a curve $C \to \operatorname{Spec} A$ then

$$\mathrm{Aut}_\xi(M) = H^0(C, T_C \otimes M)\ ,\ \mathrm{Def}_\xi(M) = H^1(C, T_C \otimes M) \text{ and } \mathrm{Ob}_\xi(M) = H^2(C, T_C \otimes M) = 0.$$

References

Artin Algebraization and quotient stacks, Jarod Alper

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