Goal.

1. desingularization $\Rightarrow$ Artin's approximation & algebraization

2. algebraic stacks with linearly reductive stabilizers at closed points are étale locally quotient stacks.

Local structure of algebraic stacks.

Artin algebraization and quotient stacks

RMK. If $A \to B$ is of finite type then geom. regular $\Leftrightarrow$ smooth.

1. Artin approximation.

k alg. closed field

Def. $A \to B$ ring map of Noeth. rings is called geometrically regular if it is flat and for every prime $\mathfrak{p} \subset A$ and every finite field extension $\Leftrightarrow$ f.g. ext. $\Updownarrow$ $\Updownarrow$ finite purely insep. ext.

$\kappa(\mathfrak{p}) \to k'$ the algebra $B \otimes_A k'$ is regular.

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Thm. (Neron - Popescu desingularization) (07GC)

Let $A \to B$ be a ring map of Noeth. rings, then it is geom. regular $\Leftrightarrow$ $B = \varinjlim B_\lambda$ is direct lim of smooth A-algebras.

Ex. $\ell \to \ell^s$ is geom. regular.

$\ell \to \bar{\ell}$ is not geom. regular if ℓ non-perfect

geom. regular if ℓ perfect.

07PX

Thm. If S is a scheme of finite type over k and $s \in S$ is a k-point, then $\mathcal{O}_{S,s} \longrightarrow \widehat{\mathcal{O}_{S,s}}$ is geom. regular.

A local ring A is called G-ring if $A \to \hat{A}$ geom. regular.

Thm (Artin approximation)

Let S be a scheme of finite type over k and $F : Sch/S \longrightarrow Sets$ be a lim preserving contravariant functor.

$\varinjlim F(\mathrm{Spec} B_\lambda) = F(\mathrm{Spec} \varinjlim B_\lambda)$

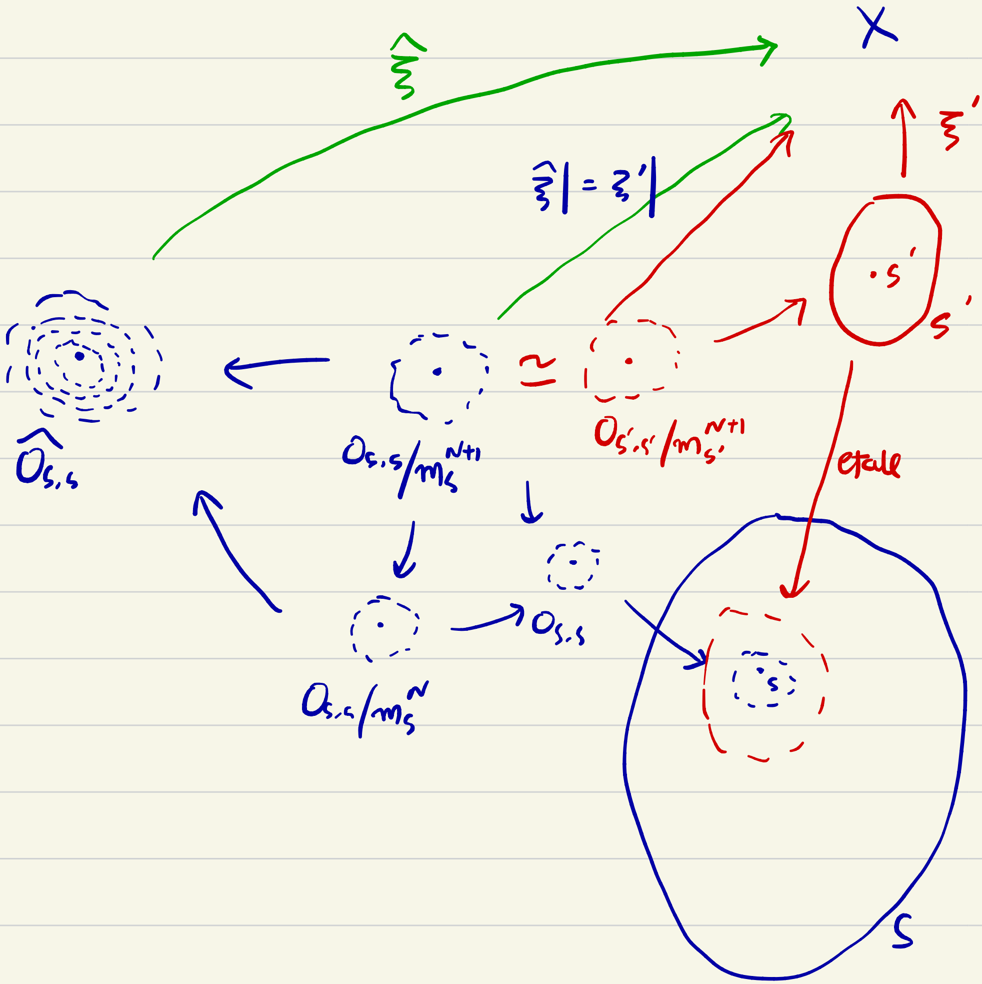
h_X lim perserving $\Leftrightarrow$ $X \to S$ LFP.

Let $s \in S$ be a k-point and $\hat{\xi} \in F(\widehat{\mathcal{O}_{S,s}})$.

Then for any $N \geq 0$, there exists an etale morphism $(S', s') \longrightarrow (S, s)$ and $\xi' \in F(S')$ s.t.

$$\hat{\xi}\big|_{\mathcal{O}_{S,s}/m_s^{N+1}} = \xi'\big|_{\mathcal{O}_{S,s}/m_s^{N+1}}$$

RMK. True for S excellent.



RMK. Not possible to find ξ' s.t. $\xi'|_{\mathcal{O}_{S,s}/m_s^{n+1}} = \hat{\xi}|_{\cdot}$, $\forall n$.

Alternative formulation.

(A, m) local Henselian G-ring

Cor. Let A f.g. k-alg., $m \subset A$ maximal ideal.

$$f_1, \dots, f_m \in A[x_1, \dots, x_n]$$

$\hat{a} \in \hat{A}^n$ ...

$\hat{a} \in \hat{A}_m^n$ s.t. $f_i(\hat{a}) = 0$, $\forall i$.

Then $\forall N \geq 0$, $\exists$ $(A, m) \longrightarrow (A', m')$ étale

$a' \in A'^n$ s.t. $a' \equiv \hat{a}$ mod m^{N+1}.

$a' \in A^n$ ...

RMK. This actually implies Artin approximation.

$X \xrightarrow{f} Y$ morphism of schemes of finite type $/k$, $x \in X$ k-point

then f etale at $x \iff f$ induces isom. on completion.

Cor. Let X_1, X_2 be schemes of finite type over k.

Suppose $x_1 \in X_1$, $x_2 \in X_2$ are k-points s.t.

$\widehat{\mathcal{O}_{X_1,x_1}} \simeq \widehat{\mathcal{O}_{X_2,x_2}}$. Then there exists a

common etale nbhd

$$\begin{tikzcd} & (X', x') \arrow[dl] \arrow[dr] & \\ (X_1, x_1) & & (X_2, x_2) \end{tikzcd}$$

Pf. $F: Sch/X_1 \to Sets$, $(T \to X_1) \longmapsto Hom(T, X_2)$

$$= Hom_{X_1}(T, X_1 \times X_2)$$

$X_1 \times X_2 \to X_1$ finite type $\Rightarrow F$ lim perserving

$\widehat{\mathcal{O}_{X_1,x_1}} \simeq \widehat{\mathcal{O}_{X_2,x_2}}$ gives an element in $F(\widehat{\mathcal{O}_{X_1,x_1}})$

$N=1 \Rightarrow \exists\ (X', x')$ etale over (X_1, x_1)

with a morphism $(X', x') \to (X_2, x_2)$

inducing Isom. $\mathcal{O}_{X_2,x_2}/m_{x_2}^2 \longrightarrow \mathcal{O}_{X',x'}/m_{x'}^2$.

Want: $\widehat{\mathcal{O}_{X_2,x_2}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X',x'}}$.

Identify $\widehat{\mathcal{O}_{X',x'}} \simeq \widehat{\mathcal{O}_{X_1,x_1}} \simeq \widehat{\mathcal{O}_{X_2,x_2}}$

ETS (A, m) local Noetherian, $\phi: A \to A$ local inducing

$A/m^2 \xrightarrow{\sim} A/m^2$, then ϕ Isom.

NAK $\Rightarrow$ ϕ Isom.

Then $(X', x') \longrightarrow (X_2, x_2)$ etale at x', hence

etale in an open nbhd. //

Def. A cat. fibered in groupoids over S is a cat. $\mathcal{X}$ together with $p: \mathcal{X} \to Sch/S$ s.t.

(1)
$$\begin{tikzcd}
\exists\, \xi_1 \arrow[r, dashed, "u"] \arrow[d, dashed, mapsto] & \xi_2 \arrow[d, mapsto] \\
T_1 \arrow[r, "f"] & T_2
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
\xi_1 \arrow[r, dashed, "\exists !"] \arrow[rr, bend left] \arrow[d, mapsto] & \xi_2 \arrow[r] \arrow[d, mapsto] & \xi_3 \arrow[d, mapsto] \\
T_1 \arrow[r] \arrow[rr, bend left] & T_2 \arrow[r] & T_3
\end{tikzcd}$$

For any $T \in Sch/S$, let $\mathcal{X}(T)$ be the cat. with objs over T and morphisms over id_T.

This is a groupoid cat.

Def. A cat. $\mathcal{X}$ fibered in groupoids over S is called lim preserving if

$$\varinjlim \mathcal{X}(\operatorname{Spec} B_\lambda) \longrightarrow \mathcal{X}(\operatorname{Spec} \varinjlim B_\lambda)$$

is an equivalence of cat. for any direct lim $\{B_\lambda\}$ of $\mathcal{O}_S$-algs.

Thm. (Groupoid version of Artin approximation)

Let S be a scheme of finite type over k

$\mathcal{X}$ lim preserving cat. fibered in groupoids over S

$s \in S$ a k-point, $\hat{\xi} \in \mathcal{X}(\operatorname{Spec} \widehat{\mathcal{O}_{S,s}})$

For any $N \geq 0$ $\exists$ etale morphism $(S', s') \to (S, s)$ and $\xi' \in \mathcal{X}(S')$ s.t.

$$\hat{\xi}\Big|_{\operatorname{Spec} \mathcal{O}_{S,s}/m_s^{N+1}} \simeq \xi'\Big|_{\operatorname{Spec} \mathcal{O}_{S,s}/m_s^{N+1}}$$

Stuff missing.

geom. regularity , relation with formal smoothness and ...

proof of desingularization

proof of facts about geom. regularity

proof of Artin approximation.

2. Artin algebraization

Artin appro. $\Rightarrow$ CJ appro. $\Rightarrow$ Artin alg.

$(A, \mathfrak{m})$ local ring, M A-mod, $Gr_{\mathfrak{m}} M = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$.

Thm. (Conrad - de Jong approximation)

Let $\mathcal{X}$ be a lim preserving cat. fibered in groupoids $/k$.

Let $(R, \mathfrak{m})$ be a complete local Noeth. k-alg.

$\hat{\xi} \in \mathcal{X}(\operatorname{Spec} R)$. Then $\forall N \geq 0$, there exist

(1) $(\operatorname{Spec} A, u)$ affine scheme, finite type $/k$, k-point u

(2) $\xi_A \in \mathcal{X}(\operatorname{Spec} A)$

(3) $\alpha_{N+1} : R/\mathfrak{m}^{N+1} \xrightarrow{\sim} A/u^{N+1}$

(4) $\hat{\xi}|_{R/\mathfrak{m}^{N+1}} \xrightarrow{\sim} \xi_A|_{A/u^{N+1}}$ via α_{N+1}

(5) $Gr_{\mathfrak{m}} R \simeq Gr_u A$ of graded k-alg.

Def. (A, m) local Noeth. $\varphi: M \to N$ between finite A-mod.

Let $c \geq 0$ integer, we say $(AR)_c$ holds for φ if

$$\varphi(M) \cap m^n N \subset \varphi(m^{n-c} M), \quad \forall\, n \geq c.$$

Artin algebraization.

Def. $\mathcal{X}$ cat. fibered in groupoids $/k$.

R complete local Noeth. k-alg. . x closed point

$\hat{\xi} \in \mathcal{X}(\operatorname{Spec} R)$ is called formally versal at x if

$$\begin{tikzcd}
\operatorname{Spec} k(x) \arrow[r] & \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{Spec} R \arrow[d, "\hat{\xi}"] \\
& \operatorname{Spec} B' \arrow[r, "\eta'"'] \arrow[ur, dashed, "\exists"] & \mathcal{X}
\end{tikzcd}$$

where $B' \twoheadrightarrow B$ surj. of artinian k-alg.

i.e. whenever $B' \twoheadrightarrow B$ surj. of artinian k-alg.

$\operatorname{Spec} B \to \operatorname{Spec} R$, $y \in \operatorname{Spec} B$ closed over x , $k(y) = k(x)$

$\eta' \in \mathcal{X}(\operatorname{Spec} B')$

$\alpha : \eta'|_{\operatorname{Spec} B} \simeq \hat{\xi}|_{\operatorname{Spec} B}$

$\exists\ \tilde{\alpha} : \operatorname{Spec} B' \to \operatorname{Spec} R$ s.t. $\circlearrowright$

and $\hat{\xi}|_{\operatorname{Spec} B'} \simeq \eta'$

extending α.

Thm. (Artin algebraization).

$\mathcal{X}$ lim preserving cat. fibered in groupoids / k

$(R, \mathfrak{m})$ complete local Noeth. k-alg.

$\hat{\xi} \in \mathcal{X}(\operatorname{Spec} R)$ formally versal. Then there exist

(1) $(\operatorname{Spec} A, u)$ finite type / k with a k-point

(2) $\xi_A \in \mathcal{X}(\operatorname{Spec} A)$

(3) $\alpha : R \xrightarrow{\sim} \widehat{A_u}$ of k-alg.

(4) compatible family of $\hat{\xi}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}} \simeq \xi_A|_{\operatorname{Spec} A/u^{n+1}}$, $\forall n$.

Def. A collection $\{S_i \to S\}$ is an etale covering if each is etale and $\coprod S_i \twoheadrightarrow S$.

Def. $\mathcal{X}/S$ is a stack over S if for any etale covering $\{T_i \to T\}$ of S-scheme T, we have

(1) morphism glue $\quad \varphi_i|_{T_{ij}} = \varphi_j|_{T_{ij}}$

$$\begin{tikzcd} & a|_{T_i} \arrow[dr] \arrow[drr, bend left, "\varphi_i"] & & \\ a|_{T_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed, "\exists !\, \varphi"'] & b \\ & a|_{T_j} \arrow[ur] \arrow[urr, bend right, "\varphi_j"'] & & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & T_i \arrow[dr] & \\ T_{ij} \arrow[ur] \arrow[dr] & & T \\ & T_j \arrow[ur] & \end{tikzcd}$$

$\varphi|_{T_i} = \varphi_i$.

(2) object glue $\quad \varphi_{ij}$ satisfy cocycle condition

$$\begin{tikzcd} & a_i \arrow[dr, dashed] & \\ a_i|_{T_{ij}} \overset{\varphi_{ij}}{\simeq} a_j|_{T_{ij}} \arrow[ur] \arrow[dr] & & \exists\, a \\ & a_j \arrow[ur, dashed] & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & T_i \arrow[dr] & \\ T_{ij} \arrow[ur] \arrow[dr] & & T \\ & T_j \arrow[ur] & \end{tikzcd}$$

$\varphi_i : a|_{T_i} \simeq a_i$ compatible with φ_{ij} on T_{ij}.

Def. $\mathfrak{X}/S$ stack is algebraic if

(1) $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable

(2) $\exists$ scheme U, representable smooth surj. $U \to \mathfrak{X}$.

2. Representability of the diagonal.

Thm. The diagonal of an algebraic space / stack is rep. by schemes / rep.

Pf. X alg. space, $U \to X$ etale presentation.

$R = U \times_X U$ scheme.

$T \to X \times X$ from a scheme, want $Q_T = X \times_{X \times X} T$ scheme.

$U \to X$ surj. etale, rep. by schemes

$\Rightarrow U \times U \to X \times X$ surj. etale, rep. by schemes

$$\begin{tikzcd}
& \overset{\text{scheme}}{Q_{T'}} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \ \text{scheme} \arrow[dl] \arrow[dd, "\text{surj. etale}"] \\
\underset{\text{scheme}}{R} \arrow[rr] \arrow[dd] & & U \times U \ \text{scheme} \arrow[dd] & \\
& Q_T \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
X \arrow[rr] & & X \times X &
\end{tikzcd}$$

Algebraic stacks and Artin's axioms.

1. Algebraic stacks

- cat. fibered in groupoids
- stacks in Sch_{et}. (2-Yoneda, fibre product)
- alg. space & alg. stacks (def., morphism, equiv. relation ...)
- formally versal

Def. $\mathcal{X}$ cat. fibered in groupoids / k, R complete local Noeth. k-alg., $x \in \operatorname{Spec} R$ closed point. $\hat{\xi} \in \mathcal{X}(\operatorname{Spec} R)$ is called formally versal at x if

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "\hat{\xi}"] \\
& \operatorname{Spec} B' \arrow[r, "\eta'"] \arrow[ur, dashed, "\exists"] & \mathcal{X}
\end{tikzcd}$$

for every $B' \to B$ surj. of Artinian k-alg.

Let $\xi_n \in \mathcal{X}(\operatorname{Spec} R/m^n)$, $f_n : \xi_n \to \xi_{n+1}$ morphism of $\mathcal{X}$ lying above $\operatorname{Spec} R/m^n \to \operatorname{Spec} R/m^{n+1}$. The formal object $(R, \{\xi_n\}, \{f_n\})$ is called (formally) versal at x if

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} R/m^n \arrow[r] \arrow[ddr, "\xi_n"] & \operatorname{Spec} R/m^{n'} \arrow[dd, "\xi_{n'}"] \\
& \operatorname{Spec} B' \arrow[urr, dashed, "\exists"] \arrow[drr, bend right, "\eta'"'] & & \\
& & & \mathcal{X}
\end{tikzcd}$$

for every $B' \twoheadrightarrow B$ surj. of (local) Artinian k-alg.

i.e. given $\xi_1 \to \eta \to \eta'$ in $\mathcal{X}$ lying over

$$\operatorname{Spec} k \to \operatorname{Spec} B \to \operatorname{Spec} B'$$

and

$$\begin{tikzcd}
& \eta \arrow[dl] \\
\xi_n & \xi_1 \arrow[l] \arrow[u]
\end{tikzcd}
\quad \text{lying over} \quad
\begin{tikzcd}
& \operatorname{Spec} B \arrow[dl] \\
\operatorname{Spec} R/m^n & \operatorname{Spec} k \arrow[l] \arrow[u]
\end{tikzcd}$$

$\exists\, n' \geq n$,

$$\begin{tikzcd}
& & \eta' \arrow[ddll] \\
& & \eta \arrow[u] \arrow[dl] \\
\xi_{n'} & \xi_n \arrow[l] & \xi_1 \arrow[l] \arrow[u]
\end{tikzcd}
\quad \text{lying over} \quad \cdots$$

- Artin's axioms.

Thm. $\mathcal{X}$ stack / k. Then $\mathcal{X}$ is alg. stack locally of finite type over k iff

(0) (lim preserving)

(1) (Rep. of diagonal)

(2) (Existence of formal deformation)

$\forall\ x : \operatorname{Spec} k \longrightarrow \mathcal{X}$, $\exists$ complete local Noeth. k-alg. (R, m), compatible family of $\xi_n \in \mathcal{X}(\operatorname{Spec} R/m^{n+1})$. $\xi_0 = x$

s.t. $(R, \{\xi_n\}, \{f_n\})$ is formally versal

(3) (effectivity) 07X8 $\Rightarrow$

$\forall\ (R, m)$ complete local Noeth. k-alg.

$$\mathcal{X}(\operatorname{Spec} R) \longrightarrow \varprojlim \mathcal{X}(\operatorname{Spec} R/m^n)$$

is an equivalence of cat.

(4) (Openness of versality)

$\forall\ \xi_U \in \mathcal{X}(U)$, U scheme of finite type $/k$, $u \in U$ s.t.

ξ_U is formally versal at u (in $\mathcal{X}(\operatorname{Spec} \widehat{\mathcal{O}_{U,u}})$).

then ξ_U is formally versal in an open nbhd of u.

Pf.

- $\overset{(1)}{\text{rep.}}$ of diagonal $\Rightarrow$ any $U \longrightarrow \mathcal{X}$ from scheme is rep.

$$\begin{tikzcd} U \times_{\mathcal{X}} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times T \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

- $\forall\ x: \operatorname{Spec} k \longrightarrow \mathcal{X}$, find

$$\begin{tikzcd} & U \arrow[d, "\text{smooth}"] & \text{scheme} \\ \operatorname{Spec} k \arrow[ur, "u"] \arrow[r, "x"'] & \mathcal{X} & \end{tikzcd}$$

(2) & (3) $\Rightarrow$ $\exists$ complete local Noeth. k-alg. (R, m)

$R/m = k$,

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[dr, "x"'] & \operatorname{Spec} R \arrow[d, "\hat{\xi}"] \\ & \mathcal{X} \end{tikzcd}$$

$\hat{\xi}$ formally versal

Artin algebraization $\Rightarrow$ $\exists$ $\operatorname{Spec} A / k$ of finite type

$u \in \operatorname{Spec} A$ k-point

$\xi_A \in \mathcal{X}(\operatorname{Spec} A)$

$\alpha : R \xrightarrow{\sim} \hat{A}_u$ of k-alg.

compatible $\hat{\xi}|_{\operatorname{Spec} R/m^{n+1}} \xrightarrow{\sim} \xi_A|_{\operatorname{Spec} A/u^{n+1}}$

$\forall n \geq 0$

$U = \operatorname{Spec} A$, $\xi_U := \xi_A \in \mathcal{X}(U)$ formally versal at u

(4) $\Rightarrow$ ξ_U is formally versal in an open nbhd of u

$\Rightarrow$ $U \to \mathcal{X}$ is smooth in an open nbhd of u

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